BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

Rédacteur en chef
K. KURATOWSKI

Rédacteur en chef suppléant
L. INFELD

SÉRIE DES SCIENCES MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

Rédacteur de la Série
A. MOSTOWSKI

Comité de Rédaction de la Série
K. BORSUK, W. IWANOWSKA, A. JABŁOŃSKI, W. RUBINOWICZ
A. SOŁTAN

VOLUME VII
NUMERO 2

VARSOVIE 1959

PRINTED IN POLAND

PAŃSTWOWE WYDAWNICTWO NAUKOWE – WARSZAWA
MIODOWA 10

Nakład 1100+150 egz.	*Rękopis dostarczono 2. I. 1959*
Ark. wyd. 4, ark. druk. 3 7/8	*Podpisano do druku 2. III. 1959*
Papier bezdrzewny sat. 80 g kl. III	*Druk ukończono w marcu 1959*
Format B5, 70×100 cm	*Zam. prod. 4/59 Cena zł 20.–*

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 2, 1959

MATHEMATICS

ε-Mappings of a Disc Onto a Torus

by

M. K. FORT, Jr.

Presented by K. KURATOWSKI on September 30, 1958

Let X and Y be metric spaces, and let ε be a positive number. An *ε-mapping* of X onto Y is a mapping (i. e. continuous function) f of X onto Y such that for each $y \in Y$, the set $f^{-1}(y)$ has diameter less than ε.

We let D be the closed unit disc in the Euclidean plane, and let T be a torus.

In The Scottish Book *), Ulam states the following problem (number 21):

Does there exist, for each $\varepsilon > 0$, an ε-mapping of D onto T?

In this paper, we show that for small ε such mappings do not exist.

LEMMA. *Let a be a simple closed curve which is the union of four arcs a_1, a_2, a_3, a_4 having at most end points in common for each pair, and satisfying $a_1 \cap a_3 = \emptyset = a_2 \cap a_4$. If K is a metric space such that $K = \bigcup_{n=1}^{4} A_n$, where each A_n is closed in K, $A_n \supset a_n$ for each n, and $A_1 \cap A_3 = \emptyset = A_2 \cap A_4$, then there exists a retraction of K onto a.*

Proof. Let h be a homeomorphism on a into the plane such that the arcs a_1, a_2, a_3, a_4 map respectively under h onto the segment from (0,0) to (0,1), the segment from (0,1) to (1,1), the segment from (1,1) to (1,0), the segment from (1,0) to (0,0).

By Urysohn's Lemma, there exist mappings φ and ψ on K into the closed interval of real numbers [0,1] such that:

$$\varphi(x) = 0 \text{ for } x \in A_1, \quad \varphi(x) = 1 \text{ for } x \in A_3,$$
$$\psi(x) = 0 \text{ for } x \in A_4, \quad \psi(x) = 1 \text{ for } x \in A_2.$$

We define a mapping $P: K \to a$ by letting

$$P(x) = h^{-1}\big((\varphi(x), \psi(x))\big).$$

*) A copy of the original Scottish Book, translated and mimeographed by S. Ulam.

It is easily seen that P maps each arc a_n onto itself, and hence $P|a$ is homotopic to the identity mapping on a. It follows from the homotopy extension theorem of Borsuk that there exists a mapping $R: K \to a$ such that $R|a$ is the identity mapping on a. R is the desired retraction.

THEOREM. *If $\varepsilon < 1/6$, then there does not exist an ε-mapping of D onto T.*

Proof. We shall give a proof by contradiction. Thus, let us assume that $0 < \varepsilon < 1/6$, and that f is an ε-mapping of D onto T.

If i is an integer, $1 \leqslant i \leqslant 6$, we define C_i to be the circle in the Euclidean plane having center at the origin and radius $i/6$. We define C_{ij} to be the intersection of C_i with the closed j-th quadrant. We define Q_j to be the closed set which consists of all points in the j-th quadrant which are between or on the circles C_1 and C_6. We define $A_j = f[Q_j]$, and observe that $A_1 \cap A_3 = \emptyset = A_2 \cap A_4$.

It is possible to choose arcs a_j such that $a_j \subset f[C_{6j}]$ for each j, $a_1 \cap a_3 = \emptyset = a_2 \cap a_4$, and $a = \bigcup_{j=1}^{4} a_j$ is a simple closed curve.

Suppose a separates T, say $T - a = T_1 \cup T_2$, with $T_1 \cap T_2 = \emptyset$ and T_1 and T_2 both open and non-empty in $T - a$. We let Δ be the disc in the Euclidean plane with boundary C_1. Since $f[\Delta] \cap a = \emptyset$, either $f[\Delta] \subset T_1$ or $f[\Delta] \subset T_2$, and we may assume without loss of generality that $f[\Delta] \subset T_1$. It follows that $T_2 \cup a \subset \bigcup_{n=1}^{4} A_n$. We may apply our lemma to obtain a retraction R of $\bigcup_{n=1}^{4} A_n$ onto a, and it follows that $R|(T_2 \cup a)$ is also a retraction onto a.

However, since $T_2 \cup a$ is either a disc with boundary a, or a disc-with-handle with boundary a, such a retraction cannot exist. Thus, we see that a does not separate T.

In the same way we obtained a, we may obtain simple closed curves β and γ, contained in $f[C_4]$ and $f[C_2]$ respectively, such that neither β nor γ separates T, and a, β, γ are pairwise disjoint.

Now, $T - (a \cup \beta \cup \gamma)$ consists of three components. One of these components, which we denote by H, will satisfy

$$(\text{boundary } H) = (a \cup \gamma)$$

and

$$(\text{closure } H) \cap \beta = \emptyset .$$

We define U to be the closed annulus bounded by C_6 and C_5, V to be the closed annulus bounded by C_5 and C_3 and W to be the closed disc bounded by C_3.

Since $f[V]$ is connected, $f[V] \supset \beta$, and $f[V] \cap (a \cup \gamma) = \emptyset$, we see that $f[V] \cap H = \emptyset$. Thus, $H^* = (H \cup a \cup \gamma) \subset f[U] \cup f[W]$. Now, $H^* \cap f[U]$

and $H^* \cap f[W]$ are disjoint non-empty compact sets, and

$$H^* = (H^* \cap f[U]) \cup (H^* \cap f[W]) .$$

Since H^* is connected, this is impossible.

Thus, for $0 < \varepsilon < 1/6$, an ε-mapping of D onto T does not exist.

The type of technique used above in solving the disc-to-torus problem has other applications. We now show how this technique can be used to obtain a result concerning closed, orientable 2-manifolds.

Two metric spaces A and B are called *quasi-homeomorphic* by Kuratowski and Ulam (see problem 97 of the Scottish Book) if for each $\varepsilon > 0$ there exists an ε-mapping of A onto B and also an ε-mapping of B onto A.

THEOREM. *If two closed, orientable, 2-manifolds A and B are quasi-homeomorphic, then they are homeomorphic.*

Proof. It is easily seen that the genus of a closed, orientable, 2-manifold M is determined by the maximum number n of simple closed curves $\mu_1, \ldots, \mu_n$ that one can obtain on M for which

$$\mu_i \cap \mu_j = \emptyset \quad \text{for} \quad i \neq j ,$$

and

$$M - \bigcup_{i=1}^{n} \mu_i \quad \text{is connected} .$$

Thus, in order to prove our theorem, it is sufficient to show that if A contains n pairwise disjoint simple closed curves $\alpha_1, \ldots, \alpha_n$ for which $A - \bigcup_{i=1}^{n} \alpha_i$ is connected, then B contains n pairwise disjoint simple closed curves $\beta_1, \ldots, \beta_n$ for which $B - \bigcup_{i=1}^{n} \beta_i$ is connected.

Suppose that $\alpha_1, \ldots, \alpha_n$ are pairwise disjoint simple closed curves on A for which $A - \bigcup_{i=1}^{n} \alpha_i$ is connected. There exists a closed cylindrical neighborhood K_i of α_i for each i such that

$$K_i \cap K_j = \emptyset \quad \text{for} \quad i \neq j ,$$

and

$$J = A - \bigcup_{i=1}^{n} K_i \quad \text{is connected} .$$

We are now able to choose $\varepsilon > 0$ small enough so that for an ε-mapping f of A onto B:

$$f[K_i] \cap f[K_j] = \emptyset \quad \text{for} \quad i \neq j ,$$

$$f^{-1} f[\alpha_i] \subset K_i \quad \text{for each } i, \text{ and}$$

each $f[\alpha_i]$ contains a simple closed curve β_i which does not separate B. (To obtain the β_i, we must use the method of the disc-torus problem).

Suppose that $L = B - \bigcup_{i=1}^{n} \beta_i$ is not connected. Then $f[J]$ is contained in a single component of L, and hence some component L_1 of L is contained in $\bigcup_{i=1}^{n} f[K_i]$. Since not one of the β_i separates B, L_1 has at least two of the β_i on its boundary. Thus, $L_1^* = (\text{closure } L_1)$ intersects at least two of the sets $f[K_i]$ non-vacuously. Since the sets $f[K_i]$ are compact and pairwise disjoint, this implies that $L_1^* = \bigcup_{i=1}^{n} (L_1^* \cap f[K_i])$ is not connected. This is a contradiction, since L_1 and hence also L_1^* is connected. Thus, $B - \bigcup_{i=1}^{n} \beta_i$ is connected.

The author is an Alfred P. Sloan Research Fellow. This research was also partially supported by the National Science Foundation Research Grant G3016.

Note of the Editor. The same result has been found independently and by means of a different method by T. Ganea and presented to the Seminar of Professor Kuratowski on May 22, 1958. It will appear in Fundamenta Mathematicae.

THE UNIVERSITY OF GEORGIA, U. S. A.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 2, 1959

MATHEMATICS

Some Remarks on the Marcinkiewicz-Orlicz Space. III

by

J. ALBRYCHT

Presented by W. ORLICZ on October 28, 1958

Lately E. Følner investigated almost-periodic functions on arbitrary infinite groups in the sense of Besicovitch. On the other hand, the present author [1] introduced what is called the Marcinkiewicz-Orlicz spaces which remain in close connection with some classes of generalized almost-periodic functions [2]. The elements of this space are classes composed of real- or complex-valued functions defined on the infinite interval.

Using Følner's results [3], we shall construct linear, normed and complete space, analogous to the Marcinkiewicz-Orlicz space, composed of classes of real- or complex-valued functions defined on arbitrary infinite groups.

The following lemma, due to Følner, is fundamental to the present considerations:

LEMMA. *Let G be an arbitrary infinite group. Then there exist denumerably many disjoint symmetric subsets* *) $E_1, E_2, \ldots$ *of G with the property that to arbitrary, finitely many elements $a_1, a_2, \ldots, a_n$ from G and an arbitrary positive integer h there exists an element $x \in G$ such that $a_i x a_j \in E_h$ for $i, j = 1, 2, \ldots, n$.*

In our infinite group G we now make a fixed choice, once and for all, of the subsets $E_1, E_2, \ldots$ in accordance with the lemma.

For an arbitrary real-valued function f on G and an arbitrary positive integer h we put according to Følner

$$\bar{S}_h f = \inf_{A,B,C} \sup_{x,y} \sum_{i=1}^{n} a_i f(x a_i y),$$

wherein

$$A = \{a_1, a_2 \ldots a_n;\ a_1, a_2, \ldots a_n\},\ a_i > 0,\ a_1 + a_2 + \ldots + a_n = 1,\ a_i \in G$$
$$B = (b_1, b_2, \ldots, b_r), \quad b_k \in G,$$
$$C = (c_1, c_2, \ldots, c_r), \quad c_k \in G,$$

and the sup is to be taken over the non-empty set of all those x, y for which all the nr elements $b_k x a_j y c_k$ are in E_h.

*) By a symmetric subset of G we mean a subset E for which $E^{-1} = E$.

Next, we put

$$\bar{S}_B f = \overline{\lim_{h\to\infty}} \bar{S}_h f .$$

Let N denote a function possessing the properties enumerated in [5], p. 93. By $E_{\bar{N}}$, we denote a set of real- or complex-valued functions g on G, such that

$$\bar{S}_B N(|g|) \leqslant 1 .$$

For any real- or complex-valued function f on G, we define the pseudo-norm by formula

$$\|f\| = \sup_{g\in E_{\bar{N}}} \bar{S}_B |fg| .$$

Next, we denote by X the space of all real- or complex-valued functions on G, and by Q a set of elements belonging to X, for which pseudo-norm is zero.

By a norm $\|\ \|$ of an element e belonging to the quotient space X/Q we mean a pseudonorm of an arbitrary representative of e.

THEOREM 1. *The quotient space X/Q is a Banach space.*

Now, we shall denote by Y the class of all real- or complex-valued functions on G, for which

$$\bar{S}_B N(|g|) < \infty .$$

A sequence $\{g_n\}_{n=1,2}$ of functions from Y is called N-fundamental, if

$$\bar{S}_B N(|g_p - g_q|) \to 0 ,$$

when p and q, independently of each other, tend to ∞ *).

THEOREM 2. *If the sequence $\{g_n\}_{n=1,2}$ of functions from Y is N-fundamental, then there exists a function g, such that*

$$\lim_{p\to\infty} \bar{S}_B N(\tfrac{1}{2}|g - y_p|) = 0 ,$$

$$\frac{g}{2} \in Y .$$

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

REFERENCES

[1] J. Albrycht, *Some remarks on the Marcinkiewicz-Orlicz Space*, Bull. Acad. Polon. Sci., Cl. III, **4** (1956), 1.

[2] — *Some remarks on the Marcinkiewicz-Orlicz Space* (II), Bull. Acad. Polon. Sci., Sér. des sci. math. astr. et phys. **7** (1959), 16.

[3] E. Følner, *Besicovitch almost periodic functions in arbitrary groups*, Mathematica Scandinavica **5** (1957), 47.

[4] J. Musielak and W. Orlicz, *Modular Spaces*, Studia Math. (in press).

[5] W. Orlicz, *Über Räume* (L^M), Bull. Acad. Polon. Sci. et Lettres, Série A (1936), 93.

*) The concept of N-fundamental sequence was first introduced in [4].

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 2, 1959

MATHÉMATIQUE

Sur certaines classes de fonctions complexes définies sur les arcs non fermés

par

W. POGORZELSKI

Présenté par T. WAŻEWSKI le 30 Octobre 1958

Dans la théorie des problèmes aux limites pour les fonctions analytiques on introduit les classes de fonctions définies sur les arcs non fermés dans le plan de la variable complexe, qui sont remarquables par leurs propriétés importantes. Dans la présente communication nous allons préciser les définitions de ces classes, leurs propriétés et quelques applications.

Soit dans le plan de la variable complexe un ensemble L de p arcs non fermés $\overset{\frown}{a_1 b_1}$, $\overset{\frown}{a_2 b_2}$, ... $\overset{\frown}{a_p b_p}$, qui n'ont pas de points communs. On admet que ces arcs ont les tangentes continues en tout point t, même aux extrémités.

Définition 1. Nous appelons classe $\mathfrak{H}_\alpha^\mu$ un ensemble de toutes les fonctions $\varphi(t)$ de la variable complexe t, définies et continues en tout point intérieur $t \in L$, qui vérifient l'inégalité

$$|\varphi(t)| < \frac{\text{const}}{\left[\prod_{\nu=1}^{p} |t-a_\nu| \cdot |t-b_\nu|\right]^\alpha} \tag{1}$$

et la condition généralisée de Hölder de la forme

$$|\varphi(t)-\varphi(t_1)| < \frac{\text{const}\,|t-t_1|^\mu}{\left[\prod_{\nu=1}^{p} |t-a_\nu| \cdot |t_1-b_\nu|\right]^{\alpha+\mu}}, \tag{2}$$

où α et μ sont des nombres réels fixés pour la classe donnée et vérifiant les inégalités:

$$0 < \mu < 1; \quad 0 \leqslant \alpha < 1; \quad \alpha + \mu < 1. \tag{3}$$

Les constantes dans les numérateurs des expressions (1) et (2) peuvent prendre pour la classe donnée toutes les valeurs positives, les points t, t_1 sont situés toujours à l'intérieur du même arc arbitraire $\widehat{a,b}$ et le point t_1 suit le point t (c. à d. $t_1 \in \widehat{tb}$).

Définition 2. Nous allons appeler respectivement classes $\mathfrak{H}_\alpha$, $\mathfrak{H}^\mu$, $\mathfrak{H}$ les ensembles suivants:

1) l'ensemble de toutes les fonctions de classe $\mathfrak{H}_\alpha^\mu$, α étant fixé et paramètre μ prenant toutes les valeurs positives satisfaisant aux conditions (3);

2) l'ensemble de toutes les fonctions de classe $\mathfrak{H}_\alpha^\mu$, μ étant fixé et paramètre α prenant toutes les valeurs satisfaisant aux conditions (3);

3) l'ensemble de toutes les fonctions de classe $\mathfrak{H}_\alpha^\mu$, si le paramètres α et μ prennent toutes les valeurs satisfaisant aux conditions (3).

Définition 3. Nous appelons transformation singulière de Cauchy une transformation qui fait correspondre à une fonction complexe $\varphi(t)$, définie sur les arcs L une fonction $\psi(t)$, définie aux points intérieurs des arcs L par l'intégrale singulière au sens de Cauchy

$$\psi(t) = \int_L \frac{\varphi(\tau)\,d\tau}{\tau - t}. \tag{4}$$

Théorème 1. *La transformation singulière de Cauchy transforme toute fonction $\varphi(\tau)$ de classe $\mathfrak{H}_\alpha^\mu$ en fonction $\psi(t)$ de la même classe si $\alpha > 0$, et en fonction de classe $\mathfrak{H}_\varepsilon^\mu$ si $\alpha = 0$, ε désigne une constante positive arbitrairement petite.*

Cette propriété est la plus importante pour les fonctions de classe $\mathfrak{H}_\alpha^\mu$. Elle résulte des théorèmes 1 et 2 que nous avons donnés dans notre travail [1] et qui restent vrais, même sous l'hypothèse que les arcs L ont les tangentes continues.

Théorème 2. *Classe $\mathfrak{H}_\alpha^\mu$ est un sous-ensemble de la classe $\mathfrak{H}_{\alpha_1}^{\mu_1}$*

$$\mathfrak{H}_\alpha^\mu \subset \mathfrak{H}_{\alpha_1}^{\mu_1} \qquad (\alpha+\mu<1;\ \alpha_1+\mu_1<1)$$

si les paramètres des classes données vérifient les inégalités

$$0 < \mu_1 \leqslant \mu; \qquad 0 \leqslant \alpha \leqslant \alpha_1. \tag{5}$$

Démonstration. Il est évident que, si une fonction $\varphi(t)$ vérifie les inégalités (1) et (2), elle vérifie à plus forte raison les inégalités qu'on obtient en augmentant la constante α. Nous avons donc

$$\mathfrak{H}_{\alpha_1}^\mu \supset \mathfrak{H}_\alpha^\mu, \qquad \text{si} \qquad \alpha < \alpha_1.$$

Pour étudier l'influence de la diminution du paramètre μ, qui se trouve au numérateur et au dénominateur dans l'inégalité (2), considérons sur tout arc $\widehat{a_\nu b_\nu}$ deux arcs arbitraires $\widehat{a_\nu T'_\nu}$ et $\widehat{T''_\nu b_\nu}$ n'ayant pas de points communs. Il faut distinguer les deux cas suivants.

1. Point t est situé à l'intérieur de l'arc $\widehat{a_\gamma T'_\gamma}$ et le point t_1 à l'intérieur de l'arc $\widehat{T''_\gamma b_\gamma}$. Alors, le module de la différence $|t-t_1|$ a une borne inférieure positive et nous concluons, d'après la limitation (1), qu'il existe une constante K, suffisamment grande pour qu'on ait

$$|\varphi(t)-\varphi(t_1)| < K \frac{|t-t_1|^{\mu_1}}{\left[\prod_{\nu=1}^{p} |t-a_\nu|\cdot|t_1-b_\nu|\right]^{\alpha+\mu_1}} \tag{6}$$

si $0 < \mu_1 < \mu$.

2. Les points t, t_1 sont tous les deux situés sur l'arc $\widehat{a_\nu T''_\nu}$, ou bien sur l'arc $\widehat{T'_\nu b_\nu}$. En admettant que $t \in \widehat{a_\nu T''_\nu}$, $t_1 \in \widehat{tT''_\nu}$ et que $|t-t_1| \leqslant |t-a_\nu|$, nous avons

$$\frac{|t-t_1|^{\mu}}{|t-a_\nu|^{\mu}} \leqslant \frac{|t-t_1|^{\mu_1}}{|t-a_\nu|^{\mu_1}}, \qquad \text{si} \qquad \mu_1 < \mu\,;$$

en outre, la distance $|t_1-b_\nu|$ a dans ce cas une borne inférieure positive et l'inégalité (6) est satisfaite, la constante K étant choisie suffisamment grande. Dans le cas $|t-t_1| > |t-a_\nu|$ on déduit sans difficulté l'inégalité (6), en égard à la limitation (1), en choisissant K suffisamment grande. Par conséquent, le théorème 2 est démontré.

Remarque. Les fonctions de classe $\mathfrak{H}_0^{\mu}$ sont d'une importance toute particulière dans les applications; ce sont notamment les fonctions $\varphi(t)$ bornées au voisinage de toute extrémité a_ν et b_ν, qui verifient une inégalité généralisée de Hölder de la forme

$$|\varphi(t)-\varphi(t_1)| < \frac{\text{const}\,|t-t_1|^{\mu}}{\left[\prod_{=1}^{p} |t-a_\nu|\cdot|t_1-b_\nu|\right]^{\mu}}\,. \tag{7}$$

D'après le théorème 2, nous avons

$$\mathfrak{H}_0^{\mu_1} \supset \mathfrak{H}_0^{\mu}, \qquad \text{si} \qquad \mu_1 < \mu\,.$$

Définition 4. Nous allons appeler une classe $\mathfrak{H}_a^{\mu}(c_{k_1}, \ldots c_{k_q})$ l'ensemble de toutes les fonctions $\varphi(t)$ de classe $\mathfrak{H}_a^{\mu}$, qui sont bornées aux voisinages des extrémités distinguées

$$c_{k_1}, c_{k_2}, \ldots c_{k_q} \qquad (q \leqslant 2p)$$

des arcs $\widehat{a_\nu b_\nu}$ (on désigne $c_{2\nu} = b_\nu$, $c_{2\nu-1} = a_\nu$, $\nu = 1, 2, \ldots p$), donc qui vérifient les inégalités suivantes

$$|\varphi(t)| < \frac{\text{const}}{\prod_{\nu=1}^{p} |t-a_\nu|^{\theta_\nu \alpha} |t-b_\nu|^{\theta'_\nu \alpha}} \tag{8}$$

$$|\varphi(t)-\varphi(t_1)| < \frac{\text{const}\,|t-t_1|^{\mu}}{\prod_{\nu=1}^{p} |t-a_\nu|^{\theta_\nu \alpha+\mu} |t_1-b_\nu|^{\theta'_\nu \alpha+\mu}} \tag{9}$$

où $\theta_\nu = 0$, si a_ν coïncide avec l'une des extrémités $c_{k_1}, \ldots c_{k_q}$, $\theta_\nu = 0$; dans le cas contraire θ'_ν prend des valeurs analogues relativement au point b_ν. On suppose bien entendu, que $\alpha > 0$, si $q < 2p$. Nous signalons qu'une fonction $\varphi(t)$ de classe $\mathfrak{H}^{\mu}_{\alpha}(c_k, \ldots, c_{kq})$ quoique bornée aux voisinages des points $c_k, \ldots, c_{kq}$, n'a pas nécessairement de limites en ces points.

Théorème 3. *Si la fonction $\varphi(\tau)$ est de classe $\mathfrak{H}^{\mu}_{\alpha}(c_{k_1}, \ldots c_{k_q})$, alors la fonction $\psi(\tau)$ transformée par la formule*

$$\psi(\tau) = \prod_{j=1}^{2p} (t-c_j)^{\alpha_j} \int_L \frac{\varphi(\tau)\,d\tau}{\prod_{j=1}^{2p} (\tau-c_j)^{\alpha_j} (\tau-t)}; \qquad t \in L \tag{10}$$

est de classe $\mathfrak{H}^{\mu}_{\varrho}(c_{k_1}, \ldots c_{k_q})$, où les constantes complexes $\alpha_j = \alpha'_j + i\alpha''_j$ vérifient les inégalités

$$0 < \alpha'_j < 1, \quad \text{si} \quad j = k_1, k_2, \ldots k_q,$$
$$-1 < \alpha'_j \leqslant 0, \quad \text{si} \quad j \neq k_1, k_2, \ldots k_q,$$

ensuite $\varrho = \alpha$, si $\alpha > \beta = \max |\alpha'_j|$, pour $j \neq k_1, \ldots k_q$ et $\varrho = \beta$, si $\alpha < \beta$; en outre on suppose que $\mu + \max |\alpha'_j| < 1$, $(j = 1, 2, \ldots 2p)$.

On démontre ce théorème en s'appuyant sur le théorème 1.

Théorème 4. *Si la fonction $\varphi(\tau)$ est de classe $\mathfrak{H}^{\mu}_{\alpha}(c_{k_1}, \ldots c_{k_q})$ alors*: 1) *toute branche de la fonction analytique*

$$\Phi(z) = \prod_{j=1}^{2p} (z-c_j)^{\alpha_j} \int_L \frac{\varphi(\tau)\,d\tau}{\prod_{j=1}^{2p} (\tau-c_j)^{\alpha_j} (\tau-z)} \tag{11}$$

(dans les mêmes hypothèses concernant les constantes α_j que précédemment) holomorphe dans le voisinage de l'extrémité distinguée c_{k_j}, coupé par l'arc correspondant $\widehat{a_\nu b_\nu}$, est bornée dans ce voisinage; 2) *toute branche holomorphe dans le voisinage d'une extrémité non distinguée $c_\nu \neq c_{k_j}$ vérifie l'inégalité*

$$|\Phi(z)| < \frac{\text{const}}{|z-c_\nu|^{\varrho}} \tag{12}$$

à la singularité faible, où le nombre ϱ est défini dans le théorème 3.

La démonstration de cette propriété, assez compliquée, sera publiée dans le travail [2].

Les fonctions définies précédemment et leurs propriétés permettent de résoudre quelques problèmes aux limites pour les fonctions analytiques et faire l'inversion de l'intégrale de Cauchy dans les hypothèses plus générales que celles contenues dans l'ouvrage de Mouskelichvili [3]. Nous ne donnerons que les solutions de deux problèmes.

Problème de Hilbert. Soit $X(z)$ la solution canonique de classe $h(c_{k_1}, \ldots, c_{k_q})$ du problème homogène de Hilbert [2]

$$X^+(t) = G(t)X^-(t), \tag{13}$$

où $G(t)$ est une fonction définie pour $t \in L$ (les extrémités comprises), différente de zéro et satisfaisant à la condition de Hölder avec l'exposant μ; on suppose que les valeurs de la fonction $G(t)$ aux extrémités distinguées $c_{k_1}, \ldots, c_{k_q}$ ne sont pas positives. Soit, en outre, une fonction donnée $g(t)$ de classe $\mathfrak{H}^{\mu}_{\alpha}(c_{k_1}, \ldots, c_{k_q})$ telle que $\alpha > \max|\alpha'_j|$ pour $j \neq k_1, k_2, \ldots k_q$, où l'on a

$$\alpha'_j + i\alpha''_j = \pm\frac{1}{2\pi i}\log G(c_j)$$

$$0 < \alpha'_j < 1, \quad \text{si} \quad j = k_1, \ldots k_q, \qquad -1 < \alpha'_j \leqslant 0, \quad \text{si} \quad j \neq k_1, \ldots k_q.$$

Dans ces suppositions, en s'appuyant sur le théorème 4 et sur les raisonnements de Mouskelichvili [3], on peut affirmer que la formule

$$\Phi(z) = \frac{1}{2\pi i}X(z)\int_L \frac{g(\tau)d\tau}{X^+(\tau)(\tau - z)} + X(z)P(z) \tag{14}$$

donne toute solution de classe $h(c_{k_1}, \ldots c_{k_q})$ du problème non homogène de Hilbert

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t) \tag{15}$$

c. à d. la fonction (14): 1) est holomorphe en dehors des coupures $\widehat{a_\nu b_\nu}$; 2) admet les valeurs limites $\Phi^+(t)$ et $\Phi^-(t)$ satisfaisant à la relation (15) en tout point intérieur $t \in L$; 3) est bornée aux voisinages des extrémités $c_{k_1}, \ldots, c_{k_q}$; 4) verifie l'inégalité à faible singularité de la forme (12) où $\varrho = \alpha$ aux voisinages des autres extrémités, $P(z)$ est une fonction entière arbitraire.

Inversion d'une intégrale de Cauchy. Soit le problème de la détermination d'une fonction $\varphi(\tau)$ vérifiant l'équation

$$\int_L \frac{\varphi(\tau)d\tau}{\tau - t} = f(t) \tag{16}$$

où $f(t)$ est une fonction définie sur L de classe $\mathfrak{H}^{\mu}_{\alpha}(c_{k_1}, \ldots, c_{k_q})$.

En s'appuyant sur le théorème 3 et sur la théorie du problème de Hilbert pour les arcs non fermés [2] on peut affirmer que:

1) si $0 \leqslant q \leqslant p$, alors toutes les solutions de classe $\mathfrak{H}(c_{k_1}, \ldots, c_{k_q})$ de l'équation (16) sont données par la formule

$$(17) \qquad \varphi(t) = -\frac{1}{\pi^2}\sqrt{\frac{R_1(t)}{R_2(t)}}\int_L \sqrt{\frac{R_2(\tau)}{R_1(\tau)}}\frac{f(\tau)\,d\tau}{\tau - t} + \frac{1}{\pi i}\sqrt{\frac{R_1(t)}{R_2(t)}}P(t)$$

où les fonctions R_1 et R_2 sont données par les formules

$$R_1(t) = \prod_{j=1}^{q}(t - c_{k_j}); \qquad R_2(t) = \prod_{\nu \neq k_j}(t - c_\nu)$$

et $P(t)$ est un polynôme arbitraire de degré $p-q-1$ au plus, ou $P = 0$ si $p = q$; ces solutions sont de classe $\mathfrak{H}_\varrho^\mu(c_{k_1}, \ldots c_{k_q})$, où

$$\varrho = \tfrac{1}{2}, \quad \text{si} \quad 0 < \alpha < \tfrac{1}{2} \quad \text{et} \quad \varrho = \alpha, \quad \text{si} \quad \tfrac{1}{2} < \alpha < 1, \quad \mu + \varrho < 1;$$

2) si $p < q \leqslant 2p$, alors la solution de l'équation (16) n'existe que sous la condition que la fonction donnée $f(t)$ vérifie les égalités

$$(18) \qquad \int_L \sqrt{\frac{R_2(t)}{R_1(t)}}\, t^\nu f(t)\,dt = 0, \qquad (\nu = 0, 1, 2, \ldots q-p-1)$$

et cette solution de classe $\mathfrak{H}_\varrho^\mu(c_{k_1}, \ldots, c_{k_q})$ est donnée par la formule (17), où il faut substituer $P = 0$.

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

OUVRAGES CITÉS

[1] W. Pogorzelski, *Sur l'équation intégrale singulière non-linéaire et sur les propriétés d'une intégrale singulière pour les arcs non-fermés*. Journ. of Mathem. and Mech. (Indiana University), **7** (1958).

[2] — *Problème généralisé de Hilbert pour les arcs non fermés*, Ann. Sc. de l'École Norm. Sup. (Paris, sous presse).

[3] N. I. Muskelishvili, *Singular integral equations*. Gröningen, Holland, 1953.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 2, 1959

MATHEMATICS

Two New Methods of Analyzing a Collection of Attributes

by

J. PERKAL

Presented by H. STEINHAUS on November 11, 1958

Let x_i $(i = 1, 2, \ldots, k)$ denote k attributes standardized so that each has a zero mean and unit variance, each of them being non-negatively correlated with each of the others. We shall call them a group of attributes. We denote by x_{ij} $(j = 1, 2, \ldots n)$ the value of attribute x_i for the individual with number j in a population (or sample) containing n individuals.

Two methods of analyzing a group of attributes of this kind are known: the single-factor method of Spearman and the multiple-factor method of Thurstone [3]. The latter method is mathematically incorrect, since it gives many solutions, is subjective and presents considerable difficulties as regards the interpretation of the factors obtained. Both methods require long and troublesome calculations.

In this paper I am presenting two new methods of analyzing a group of attributes described above, a single-vector method and a multiple-vector method. The two methods are univalent, easy and simple in calculation, giving vectors (analogues of factors) which have a clear meaning in natural sciences. However, unlike Thurstone's method, these methods give vectors that are not orthogonal, either one to another or to the residuals which are left of the initial attributes after deleting the vectors

In [1] I described the index $m_{(j)} = \frac{1}{k}\sum_{i=1}^{k} x_{ij}$ and the indices $w_{ij} = x_{ij} - m_j$. Let us denote by m the arithmetic mean of k attributes of group $m = \frac{1}{k}\sum_{i=1}^{k} x_i$. This mean is called the vector of the first order of the group of attributes in question. Teissier already drew attention to it [2]. The magnitude of that vector for the j-th individual, m_j, defines the general magnitude of the individual in question with respect to all the k attributes of the group. It is a summary attribute containing a part

of the information given by the values of the attributes of the group $x_{1j}, x_{2j}, \ldots, x_{kj}$. The rest of that information is contained in the residuals w_{ij}.

Mrs. H. Milicer drew my attention to the need and possibility of splitting the groups of attributes into appropriate subgroups.

Denote by $r_{pq} = r_{qp}$ the correlation coefficient between the attributes x_p and x_q ($r_{pp} = 1$), and by $R = \|r_{pq}\|$ the correlation matrix. Let $\bar{r}_p$ denote the arithmetic mean of the p-th column (or the p-th row) of that matrix, and $\bar{\bar{r}}$ the arithmetic mean of all the elements of the matrix R. It is easy to find the expected values E and variances D of the vector m and the residuals w_p, and also of the coefficients of correlation between those quantities.

$$Em = 0;\ Dm = \bar{\bar{r}};\ Ew_p = 0;\ Dw_p = 1 + \bar{\bar{r}} - 2\bar{r}_p,$$

$$r_{m,w_p} = (\bar{r}_p - \bar{\bar{r}})/\sqrt{\bar{\bar{r}}\,(1 + \bar{\bar{r}} - 2\bar{r}_p)},$$

$$r_{w_p,w_q} = (r_{pq} + \bar{\bar{r}} - \bar{r}_p - \bar{r}_q)/\sqrt{(1 + \bar{\bar{r}} - 2\bar{r}_p)(1 + \bar{\bar{r}} - 2\bar{r}_q)}.$$

Denote by $R_{\mathrm{II}} = \|r_{w_p,w_q}\|$ the matrix of correlation between the residuals. It is impossible for all the elements of that matrix to be positive (they could be zeros only if $2r_{pq} = \bar{r}_p + \bar{r}_q$).

If the coefficients of correlation between the residuals were negative for all $p \neq q$, I should find that the vector m characterized all common elements in a group of attributes, and the part of information contained in that group of attributes and not contained in the vector m would indeed be contained in the residuals w_p.

In the opposite case, some coefficients of correlation between the residuals would be positive and some negative. I denote the residuals (or attributes) correlated in this way as a collection of residuals (or attributes). A collection of residuals (or attributes) can be split into groups in such a way that each two residuals belonging to the same group will be non-negatively correlated. I denote them as subgroups of the 2nd order. It may happen that a certain residual of the battery is non-negatively correlated with each residual of two subgroups, i. e. it can be joined to one or the other of the two subgroups. I join it to that subgroup with which it is more strongly correlated, i. e. I find the mean coefficient of correlation between that residual and the remaining residuals of one subgroup and of the other subgroup, and join the residual to that subgroup for which the correlation coefficient is greater.

For the i-th subgroup of the 2nd order I determine the vector m_1 of the 2nd order in the same way as I determined before the vector m for a group of attributes (the residuals must first be standardized, i. e. divided by their standard deviation). Those vectors are the sums of the standardized residuals of the group, and thus are the suitably weighted

sums of the attributes of that subgroup. Consequently, they have a clear interpretation in natural sciences. Two different vectors of the 2nd order are, of course, the weighted sums of attributes belonging to disjoint subgroups.

If necessary, the procedure described above may be iterated. We must then find the residuals of the 3rd order and the coefficients of correlation between them, and then divide the residuals of the 3rd order into subgroups of the 3rd order. For each of them we can find the vector of the 3rd order. This procedure can be repeated until all the residuals of the row in question prove to be negatively correlated.

By this method a group of attributes may thus be replaced by a system of vectors (of ever higher orders) containing less and less information and the residuals of the individual attributes (also of ever higher orders) negatively correlated with the vectors and with the other residuals.

The method presented here can also be applied to a collection of attributes of which some are negatively correlated (if every attribute were negatively correlated with every other attribute, then by changing the sign of any attribute we should obtain a different situation). Namely, in the way described above a collection of such attributes can be split into groups of attributes non-negatively correlated. During this procedure some attributes (just as some residuals above) might remain outside the groups.

The method of determining the vector of a group of attributes and the method, resulting from it, of determining a system of vectors for a group or a collection of attributes have the advantages mentioned in the introduction.

An example of applying the method to antropological investigations:

Eight anthropological attributes for two individuals

	A	B
Metrical attributes in mm		
length of head $(g-op)$	181	190
breadth of head $(eu-eu)$	149	163
breadth of face $(zy-zy)$	128	151
height of face $(n-gn)$	114	119
height of nose $(n-sn)$	55	50
breadth of nose $(al-al)$	38	41
Pigmentation attributes in a suitable scale		
colour of eyes	6	12
colour of hair	S	R

Four standardized vectors of individuals

Vector of the 1st order of metric attributes	—0.69	+1.14
Vector of the 1st order of 2 pigmentation attributes	—0.84	+0.15
Vector of the 2nd order of breadth (of head and face)	—1.41	+1.27
Vector of the 2nd order of height (of face and nose)	+0.25	—1.30

Interpretation: The individual A has a head smaller than medium, light pigmentation, narrow head and face of medium height. The individual B has a large head, medium pigmentation, broad head and face, low face and nose.

CHAIR OF APPLIED MATHEMATICS, WROCŁAW UNIVERSITY
(KATEDRA ZASTOSOWAŃ MATEMATYKI UNIWERSYTETU WROCŁAWSKIEGO)

REFERENCES

[1] J. Perkal, *O wskaźnikach antropologicznych*. Przegl. Antrop. **19** (1953), 210-221.

[2] G. Teissier, *Allométrie de taille et variabilité chez Maia squinado*. Zool. Exper., **42** (1955), 221-264.

[3] L. L. Thurstone, *Multiple-factor analysis*, Chicago Press, 1947.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 2, 1959

On a Paradox in Applied Probabilities

by

H. STEINHAUS and S. TRYBUŁA

Presented by H. STEINHAUS on December 6, 1958

There are many instances in engineering where we are concerned with comparing two parameters without determining their respective values. A comparison procedure is often sufficient for practical purposes, and it is simpler than determining both the parameters in question.

To fix ideas, let us suppose that we have two iron bars of the same size and shape, and that their extremities are held by a frame in such a manner that we can apply the same force to both by turning a screw. It is assumed that the screw acts on the mid-points of the bars, the force exerted being perpendicular to both bars.

We now perform the following experiment: the screw is turned until one of the bars breaks. We denote by a the resistance of the first bar to break, and denote by b the resistance of the second bar to break. The experiment is a proof for $a < b$.

If we are to compare two lots of such bars, say I and II, we repeat several times the experiment described above, choosing each time one bar from I and one from II. Let x and y denote the resistances of bars from lots I and II, respectively; then x and y are independent random variables. Let us suppose that in 60% of the experiments a bar from lot I broke first. We can then write

$$\Pr\{x < y\} = 0.6\,. \tag{1}$$

Let us now consider a third lot, denoted by III, and let z denote the resistance of bars from this lot. It can happen that a statistical comparison of lot II with lot III yields the result

$$\Pr\{y < z\} = 0.6\,. \tag{2}$$

In view of the above, (1) qualifies lot I as worse than lot II, and (2) qualifies lot II as worse than lot III. To check this result we compare III

with I by the same method as described above. Should we get

$$\Pr\{z < x\} = 0.6\,, \tag{3}$$

we would be tempted to explain the paradox expressed by (1) (2) (3) by the "error of sampling", i. e., by the fact that a finite number of experiments is not sufficient to give reliable information about lots I, II and III.

It is, however, easy to refute this explanation by defining three independent random variables x, y, z which satisfy (1) (2) (3).

It can be shown that $\sqrt{5}-1/2 (= 0.618..)$ is the greatest value of t in

$$\Pr\{x < y\} = \Pr\{y < z\} = \Pr\{z < x\} = t\,, \tag{4}$$

whatever the independent random variables x, y and z.

Thus, it would be impossible to replace in our example (expressed by (1)-(3)) the number 0.6 by 0.62.

The mathematical problem involved in the above considerations can be described as follows:

Let $F(\alpha), G(\alpha), H(\alpha)$ denote the distribution functions of the independent random variables x, y, z respectively; if we assume that no two such functions have common points of discontinuity, we get

$$\Pr\{x < y\} = \xi, \quad \Pr\{y < z\} = \eta, \quad \Pr\{z < x\} = \zeta\,,$$

with

$$\xi = \int_{-\infty}^{\infty} F(\alpha)\,dG(\alpha), \quad \eta = \int_{-\infty}^{\infty} G(\alpha)\,dH(\alpha), \quad \zeta = \int_{-\infty}^{\infty} H(\alpha)\,dF(\alpha)\,. \tag{5}$$

Thus, we have to determine which points (ξ, η, ζ) can be obtained from (5) by all possible distribution functions F, G, and H. We have the following

THEOREM. *The set of points in question is defined by*

$$1-\varrho(1-\xi, 1-\eta) \leqslant \zeta \leqslant \varrho(\xi, \eta)\,, \tag{6}$$

where

$$\varrho(\xi, \eta) = \begin{cases} \max\left(1-\xi\eta, \dfrac{1-\xi}{\eta}, \dfrac{1-\eta}{\xi}\right), & \text{for} \quad \xi+\eta > 1 \\ 1, \quad \text{for} \quad \xi+\eta \leqslant 1\,, \end{cases},$$

and $0 \leqslant \xi \leqslant 1,\ 0 \leqslant \eta \leqslant 1$.

This theorem implies for t in (4)

$$\max t = \frac{\sqrt{5}-1}{2}, \quad \min t = \frac{3-\sqrt{5}}{2}\,. \tag{7}$$

It also implies $\xi\eta\zeta \leqslant 1/4$. The inequality $\xi+\eta+\zeta \leqslant 2$ follows easily (without the Theorem) from the elementary probability theory.

EXAMPLE. Let c be a constant ($c > 4$), and let us define the independent random variables x, y, z by

$$\Pr\{x = c - \tfrac{5}{3}\} = \tfrac{3}{8}, \quad \Pr\{x = c + 1\} = \tfrac{5}{8},$$

$$\Pr\{y = c - 1\} = \tfrac{5}{8}, \quad \Pr\{y = c + \tfrac{5}{3}\} = \tfrac{3}{8}, \tag{8}$$

$$\Pr z = c - \tfrac{2}{3}\sqrt{30} = \tfrac{1}{16}, \quad \Pr\{z = c\} = \tfrac{7}{8}, \quad \Pr\{z = c + \tfrac{2}{3}\sqrt{30}\} = \tfrac{1}{16}.$$

The above definitions imply $x > 0$, $y > 0$, $z > 0$, and also

$$\Pr\{x < y\} = \Pr\{y < z\} = \Pr\{z < x\} = \tfrac{3}{6}\tfrac{9}{4} > 0.6. \tag{9}$$

Moreover, we get easily from (8)

$$E(x) = E(y) = E(z) = c, \tag{10}$$

and

$$\omega(x) = \omega(y) = \omega(z) = \tfrac{5}{3}, \tag{11}$$

where $E(v)$ reads as the expected value of v, and $\omega(v)$ as the variance of v.

The example given above shows that the information given by the usual statistics, i. e. the expected value and variance, does not show the way out of the circular inequalities (9).

Details will be given in a paper to appear in the quarterly Zastosowania Matematyki.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 2, 1959

MATHÉMATIQUE

Sur l'existence des solutions périodiques de l'équation différentielle du second ordre

par

Z. OPIAL

Présenté par T. WAŻEWSKI le 18 Décembre 1958

1. Considérons l'équation différentielle du second ordre

$$u'' = f(t, u, u') . \tag{1}$$

Relativement au deuxième membre de cette équation admettons l'hypothèse suivante:

HYPOTHÈSE (H). *La fonction $f(t, u, v)$ est continue dans tout l'espace (t, u, v) et périodique de période* 1 *par rapport à t. Pour tout triple de nombres (t_0, u_0, v_0) il existe une solution, et une seulement, $u(t)$ de l'équation* (1) *telle que $u(t_0) = u_0$ et $u'(t_0) = v_0$.*

D'après le théorème bien connu de J. Massera ([1], Th. 2), appliqué à l'équation du second ordre, l'équation (1) admet au moins une solution périodique de période 1, si toutes les intégrales de (1) sont définies pour tout $t \geqslant 0$ et s'il en existe une au moins qui est bornée avec sa dérivée dans l'intervalle $[0, +\infty)$. Le but de la présente note est d'établir un simple théorème d'existence des solutions périodiques de l'équation (1) dans un cas où les hypothèses de J. Massera ne sont plus vérifiées.

2. THÉORÈME 1. *Soit $f(t, u, v)$ une fonction croissante au sens large par rapport à u et satisfaisant à l'hypothèse* (H). *Dans ces conditions, si l'équation* (1) *admet une solution bornée avec sa dérivée pour $t \geqslant 0$, elle admet aussi au moins une solution périodique de période* 1.

3. Dans la démonstration de ce théorème nous aurons recours à deux lemmes.

LEMME 1. *Soit $f(t, u, v)$ une fonction continue, croissante au sens large par rapport à u. Admettons, de plus, l'unicité des solutions de l'équation* (1) *et soient $u(t)$, $v(t)$ deux solutions de* (1) *définies pour tout t. Ceci posé, trois cas sont possibles:*

1° *la fonction $u(t) - v(t)$ est monotone dans tout l'intervalle $(-\infty, +\infty)$;*

2^0 *$u(t)-v(t)$ est positive dans $(-\infty, +\infty)$ et il existe un t_0 tel, que cette fonction est non-décroissante dans $(t_0, +\infty)$ et non-croissante dans $(-\infty, t_0)$;*

3^0 *la fonction $u(t)-v(t)$ est négative dans $(-\infty, +\infty)$ et il existe un t_0 tel, qu'elle est non-croissante dans $(t_0, +\infty)$ et non-décroissante dans $(-\infty, t_0)$.*

Dans le cas où la fonction $f(t, u, v)$ est croissante au sens strict l'hypothèse de l'unicité est superflue.

Démonstration. Supposons d'abord que $f(t, u, v)$ soit croissante au sens strict. Pour démontrer notre lemme il suffit de démontrer que $u(t)-v(t)$ ne peut admettre ni un maximum positif, ni un minimum négatif. Or, si l'on a pour un t_0:

$$u(t_0) > v(t_0) \big(u(t_0) < v(t_0)\big) \quad \text{et} \quad u'(t_0) = v'(t_0),$$

alors par l'hypothèse

$$u''(t_0)-v''(t_0) = f\big(t_0, u(t_0), u'(t_0)\big) - f\big(t_0, v(t_0), v'(t_0)\big) > 0 \quad (< 0).$$

Dans le cas, où $f(t, u, v)$ est une fonction croissante au sens large, on peut introduire une suite d'équations auxiliaires

$$(1_n) \qquad u'' = f(t, u, u') + \frac{1}{n} u$$

aux deuxièmes membres croissants au sens strict et considérer $u(t)$ et $v(t)$ comme limites (uniformes dans tout intervalle fini) de deux suites $\{u_n(t)\}$, $\{v_n(t)\}$ convenablement choisies d'intégrales des équations (1_n).

Du lemme précédent on obtient immédiatement le suivant

Lemme 2. *Dans les hypothèses du lemme 1 les limites (finies ou non):*

$$\lim_{t\to-\infty} \big(u(t)-v(t)\big), \qquad \lim_{t\to+\infty} \big(u(t)-v(t)\big)$$

existent et au moins l'une d'elles est différente de zéro.

Remarque. Le choix convenable d'une suite $\{u_n(t)\}$ de solutions des équations (1_n) telle que l'on ait $\lim_{n\to\infty} u_n(t) = u(t)$ n'est possible, en général, que dans le cas de l'unicité des solutions de l'équation (1). Il est facile de montrer que sans l'hypothèse de l'unicité le lemme 1 n'est plus exact. En effet, l'équation

$$u'' = -2\sqrt{|u'|}$$

admet les solutions suivantes:

$$u(t) \equiv 0 \quad \text{et} \quad v(t) = 1 - \tfrac{1}{3}|t|^3.$$

4. Démonstration du théorème 1. Soit $u(t)$ une solution de (1) bornée avec sa dérivée dans l'intervalle $[0, +\infty)$:

$$|u(t)| \leqslant M, \qquad |u'(t)| \leqslant M \quad \text{pour} \quad t \geqslant 0.$$

Les suites des fonctions $u_n(t) = u(t+n)$, $u'_n(t) = u'(t+n)$ forment les familles normales dans tout intervalle fini. On peut donc trouver une suite d'indices $\{n_k\}$ telle, que la suite de fonctions $\{u_{n_k}(t)\}$ soit convergente dans l'intervalle $(-\infty, +\infty)$ vers une solution $v(t)$ de l'équation (1) telle, que

$$|v(t)| \leqslant M, \quad |v'(t)| \leqslant M \quad -\infty < t < +\infty. \tag{2}$$

Désignons par E la famille de toutes les solutions de l'équation (1) qui vérifient les inégalités (2) et posons pour tout $u(t) \in E$:

$$G_{u(t)} = \operatorname*{borne\,sup}_{-\infty<t<+\infty} u(t), \qquad g_{u(t)} = \operatorname*{borne\,inf}_{-\infty<t<+\infty} u(t).$$

On vérifie sans peine que E est un ensemble compact au sens de la convergence presque uniforme dans l'intervalle $(-\infty, +\infty)$. Donc, le sous-ensemble F de fonctions $u(t) \in E$ pour lesquelles l'oscillation

$$O_{u(t)} = G_{u(t)} - g_{u(t)}$$

acquiert la valeur la plus petite possible, n'est pas vide. De même, le sous-ensemble H de fonctions $u(t) \in F$, pour lesquelles la borne $g_{u(t}$ est la plus petite possible, contient au moins un élément.

On démontre facilement que pour tout $u(t) \in H$:

$$\limsup_{t\to-\infty} u(t) = \limsup_{t\to+\infty} u(t) = G_{u(t)}, \tag{3}$$

$$\liminf_{t\to-\infty} u(t) = \liminf_{t\to+\infty} u(t) = g_{u(t)}. \tag{4}$$

En effet, si l'on avait, par exemple, $\limsup_{t\to+\infty} u(t) < G_{u(t)}$ de la suite $u_n(t) = u(t+n)$ il serait possible d'extraire une suite partielle convergente vers une solution $v(t)$ de l'équation (1), appartenant à l'ensemble H et pour laquelle on aurait

$$G_{v(t)} - g_{v(t)} < G_{u(t)} - g_{u(t)},$$

ce qui est en contradiction avec la définition de l'ensemble F. Il résulte des relations (3) et (4) que l'ensemble H contient un seul élément $u(t)$. En effet, d'après le lemme 2, on aurait pour tout autre $v(t) \in H$ au moins une des inégalités

$$\lim_{t\to-\infty} \big(u(t) - v(t)\big) \neq 0, \qquad \lim_{t\to+\infty} \big(u(t) - v(t)\big) \neq 0,$$

ce qui est incompatible avec les relations (3), (4) et

$$g_{v(t)} = g_{u(t)} \quad \text{et} \quad G_{v(t)} = G_{u(t)}.$$

L'élément unique de l'ensemble H sera dit dans la suite solution à l'oscillation minimale ou tout simplement solution minimale de l'équation (1).

La solution minimale $u(t)$ de l'équation (1) est une fonction périodique de période 1. En effet, il est évident que la fonction $u(t+1)$ est la solution minimale de l'équation

$$u'' = f(t+1, u, u'),$$

c'est-à-dire de l'équation (1), d'où il résulte que $u(t) = u(t+1)$. Donc, le théorème 1 se trouve ainsi démontré.

5. Soit $v(t)$ une autre solution de l'équation (1), bornée dans l'intervalle $(-\infty, +\infty)$. D'après le lemme 2 ils existent des limites finies

$$\lim_{t\to-\infty}\big(v(t)-u(t)\big) = c, \qquad \lim_{t\to+\infty}\big(v(t)-u(t)\big) = C. \tag{5}$$

De même, on a

$$\lim_{t\to-\infty}\big(v(t+1)-u(t+1)\big) = c, \qquad \lim_{t\to+\infty}\big(v(t+1)-u(t+1)\big) = C. \tag{6}$$

La solution minimale $u(t)$ étant périodique, il en résulte que

$$\lim_{t\to-\infty}\big(v(t+1)-v(t)\big) = \lim_{t\to+\infty}\big(v(t+1)-v(t)\big) = 0,$$

ce qui, d'après le lemme 2, n'est possible que dans le cas où $v(t) = v(t+1)$. Il en vient que $c = C$ et, par suite, en vertu du même lemme, $v(t) = u(t)+C$.

Soit maintenant $w(t)$ une solution de l'équation (1) bornée pour $t \geqslant 0$. D'après le lemme 2 on a

$$\lim_{t\to+\infty}\big(w(t)-u(t)\big) = c.$$

Il est facile de démontrer que la fonction $u(t)+c$ est alors une solution de l'équation (1). Si la fonction $f(t, u, v)$ est croissante au sens strict, ce n'est possible que pour $c = 0$.

De toutes ces remarques on obtient le suivant

Théorème 2. *Dans les hypothèses du théorème* 1 *toute solution bornée dans l'intervalle* $(-\infty, +\infty)$ *est périodique de période* 1 *et ne diffère que d'une constante de toute autre solution périodique de l'équation* (1).

Pour toute solution $v(t)$ *de l'équation* (1), *bornée pour* $t \geqslant 0$, *il existe une solution périodique de cette équation* $u(t)$ *telle, que* $\lim_{t\to+\infty}\big(v(t)-u(t)\big) = 0$. *Il en est de même des solutions bornées pour* $t \leqslant 0$.

Si la fonction $f(t, u, v)$ *est croissante au sens strict, l'équation* (1) *admet une solution périodique, et une seulement.*

6. Dans l'énoncé du théorème 1 on suppose qu'une solution $u(t)$ de l'équation (1) est bornée avec sa dérivée $u'(t)$ dans l'intervalle $(0, +\infty)$. Cependant, il suffit quelquefois de supposer que l'équation (1) admet une solution $u(t)$ bornée dans l'intervalle $(0, +\infty)$, car le fait que la

dérivée $u'(t)$ est, elle aussi, bornée dans le même intervalle en est une simple conséquence.

Il en est ainsi, par exemple, dans le cas où la fonction $f(t, u, v)$ satisfait à l'inégalité

$$|f(t, u, v)| \leqslant M(|u|)\cdot V(|v|)\,,$$

où les fonction $M(u)$ et $V(u)$ sont continues, positives et où, de plus,

$$\int_0^\infty \frac{v\,dv}{V(v)} = +\infty \qquad V(v) \geqslant m > 0\,.$$

Ce théorème résulte immédiatement de la démonstration du théorème II de notre travail [2] (§ 2, pp. 318-321).

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

OUVRAGES CITÉS

[1] J. Massera, *The existence of periodic solutions of systems of differential equations*, Duke Math. Journal **17** (1950), 457-475.

[2] Z. Opial, *Sur les intégrales bornées de l'équation* $u'' = f(t, u, u')$. Ann. Pol. Math. **4** (1958), 314-324.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 2, 1959

On the Lee Model

by

I. BIAŁYNICKI-BIRULA

Presented by L. INFELD on November 29, 1958

Introduction

The solvable model of the quantum field theory introduced in 1954 by T. D. Lee [1] has appeared very interesting from the theoretical point of view. In this model the renormalization process leads to the indefinite metric in the space of state vectors, as was shown by W. Pauli [2] and W. Heisenberg [3]. However, the space of state vectors with indefinite metric has not as yet a physical interpretation. Therefore, it is very interesting to answer the question whether it is necessary to introduce the indefinite metric also in the usual local theory. It follows from our paper that the Lee model at any rate cannot answer this question.

In this paper we examine the model of the local quantum field theory which describes two fermion fields (the fermions are infinitely heavy) in the interaction with the boson field *). In such a model there is no difficulty connected with the renormalization. The unrenormalized and renormalized coupling constants are equal. The Lee model corresponding to such a theory has been investigated by Källen and Pauli [3]. In this model there appear difficulties characteristic for the Lee model. Hence, it follows that the necessity of introducing the indefinite metric is caused by the simplifications made in the Lee model. These simplifications, as is well known, mean the removal of half of the terms from the interaction Hamiltonian.

The lack of these terms in Lee's interaction Hamiltonian disturbs the symmetry between the creation and the annihilation of bosons and leads to nonlocality of the Lee model.

*) The model of the field theory with heavy fermions was investigated, among others, by Tomanaga [4], Wentzel [5], van Hove [6], [7], Greenberg and Schweber [8].

Model of the local theory

We denote the fermion fields by $V(\boldsymbol{x})$ and $N(\boldsymbol{x})$. We shall not take into account the spin of fermions and their antiparticles. We shall denote the boson field and the canonical conjugated field by $\Phi(\boldsymbol{x})$ and $\Pi(\boldsymbol{x})$, respectively.

$$(1)\qquad V(\boldsymbol{x})=\frac{1}{(2\pi)^{3/2}}\int d^3p V(\boldsymbol{p})e^{i\boldsymbol{p}\boldsymbol{x}},\qquad N(\boldsymbol{x})=\frac{1}{(2\pi)^{3/2}}\int d^3p N(\boldsymbol{p})e^{i\boldsymbol{p}\boldsymbol{x}},$$

$$\Phi(\boldsymbol{x})=\frac{1}{(2\pi)^{3/2}}\int d^3k\frac{f(\omega)}{\sqrt{2\omega}}[a(\boldsymbol{k})e^{i\boldsymbol{k}\boldsymbol{x}}+a^*(\boldsymbol{k})e^{-i\boldsymbol{k}\boldsymbol{x}}],$$

$$\Pi(\boldsymbol{x})=\frac{1}{(2\pi)^{3/2}}\frac{1}{i}\int f(\omega)\sqrt{\frac{\omega}{2}}[a(\boldsymbol{k})e^{i\boldsymbol{k}\boldsymbol{x}}-a^*(\boldsymbol{k})e^{-i\boldsymbol{k}\boldsymbol{x}}],$$

$$(2)\qquad \{V(\boldsymbol{p}),V^*(\boldsymbol{p}')\}=\delta(\boldsymbol{p}-\boldsymbol{p}'),\qquad \{N(\boldsymbol{p}),N^*(\boldsymbol{p}')\}=\delta(\boldsymbol{p}-\boldsymbol{p}'),$$

$$[a(\boldsymbol{k}),a^*(\boldsymbol{k}')]=\delta(\boldsymbol{k}-\boldsymbol{k}').$$

We introduce the notation:

$$(3)\qquad \psi(\boldsymbol{x})=\begin{Vmatrix} V(\boldsymbol{x})\\ N(\boldsymbol{x})\end{Vmatrix},\qquad \psi^*(\boldsymbol{x})=\|V^*(\boldsymbol{x}),N^*(\boldsymbol{x})\|.$$

We take the Hamiltonian for this model in the form:

$$(4)\qquad H=H_0+H_{\mathrm{I}}$$

$$H_0=m_0\int d^3pV^*(\boldsymbol{p})V(\boldsymbol{p})+m_0\int d^3pN^*(\boldsymbol{p})N(\boldsymbol{p})+\int d^3k\omega(\boldsymbol{k})a^*(\boldsymbol{k})a(\boldsymbol{k}).$$

$$(5)\qquad H_{\mathrm{I}}=g_0\int d^3xj(\boldsymbol{x})\Phi(\boldsymbol{x})=g_0\int d^3kj(\boldsymbol{k})\Phi(\boldsymbol{k})=$$

$$=\frac{g_0}{(2\pi)^{3/2}}\int d^3p\,d^3k\frac{f(\omega)}{\sqrt{2\omega}}[V^*(\boldsymbol{p})N(\boldsymbol{p}-\boldsymbol{k})a(\boldsymbol{k})+N^*(\boldsymbol{p})V(\boldsymbol{p}+\boldsymbol{k})a^*(\boldsymbol{k})+$$

$$+N^*(\boldsymbol{p})V(\boldsymbol{p}-\boldsymbol{k})a(\boldsymbol{k})+V^*(\boldsymbol{p})N(\boldsymbol{p}+\boldsymbol{k})a^*(\boldsymbol{k})],$$

where

$$j(\boldsymbol{x})=\psi^*(\boldsymbol{x})\sigma_1\psi(\boldsymbol{x})=N^*(\boldsymbol{x})V(\boldsymbol{x})+V^*(\boldsymbol{x})N(\boldsymbol{x}).$$

In order to find the exact solutions of the Schrödinger equation we perform the unitary transformation *):

$$(6)\qquad U=\exp\left(ig_0\int d^3x\frac{1}{\omega^2}\pi(\boldsymbol{x})j(\boldsymbol{x})\right)=$$

$$=\exp\left\{ig_0\int d^3k\frac{f(\omega)}{\sqrt{2\omega^3}}\frac{1}{i}[a(\boldsymbol{k})j^*(\boldsymbol{k})-a^*(\boldsymbol{k})j(\boldsymbol{k})]\right\}.$$

*) It is the clothing transformation of Greenberg and Schweber [8].

After this transformation the Hamiltonian takes the form:

$$(7)\quad \tilde{H} = UHU^{-1} = m\int d^3p\psi^*(\boldsymbol{p})\psi(p) + \int d^3k\omega(\boldsymbol{k})a^*(\boldsymbol{k})a(\boldsymbol{k}) + \\ -\frac{g_0^2}{2}\int d^3k\frac{f^2(\omega)}{\omega^2}:j^*(\boldsymbol{k})j(\boldsymbol{k}): = \bar{H}_0 + H_{\mathrm{II}}$$

$$m = m_0 - \delta m = m_0 - g_0^2\int d^3k\frac{f^2(\omega)}{\omega^2}\,.$$

$$(8)\quad H_{\mathrm{II}} = -\frac{g_0^2}{2}\int d^3k\frac{f^2(\omega)}{\omega^2}:j^*(\boldsymbol{k})j(\boldsymbol{k}): = -\frac{g_0^2}{2}\int d^3x_1d^3x_2K(\boldsymbol{x}_1-\boldsymbol{x}_2):j(\boldsymbol{x}_1)j(\boldsymbol{x}_2):$$

$$(9)\quad = +\frac{g_0^2}{2}\int d^3k\frac{f^2}{\omega^2}d^3p_1d^3p_2[V^*(\boldsymbol{p}_1)V^*(\boldsymbol{p}_2)N(\boldsymbol{p}_1-\boldsymbol{k})N(\boldsymbol{p}_2+\boldsymbol{k}) + \\ +N^*(\boldsymbol{p}_1)N^*(\boldsymbol{p}_2)V(\boldsymbol{p}_1-\boldsymbol{k})V(\boldsymbol{p}_2+\boldsymbol{k}) + V^*(\boldsymbol{p}_1)N^*(\boldsymbol{p}_2)N(\boldsymbol{p}_1-\boldsymbol{k})V(\boldsymbol{p}_2+\boldsymbol{k}) + \\ +N^*(\boldsymbol{p}_1)V^*(\boldsymbol{p}_2)V(\boldsymbol{p}_1-\boldsymbol{k})N(\boldsymbol{p}_2+\boldsymbol{k})]\,.$$

The function $K(\boldsymbol{x}_1-\boldsymbol{x}_2) = \dfrac{1}{(2\pi)^3}\int d_3k\dfrac{f^2(\omega)}{\omega^2}e^{ik(x_1-x_2)}$ describes the interaction between pairs of fermions. The simplest eigenstates of the Hamiltonian $\tilde{H}$ have the form:

$$(10)\quad \tilde{H}|0\rangle = 0\,,\quad \tilde{H}V^*(\boldsymbol{p})|0\rangle = mV^*(\boldsymbol{p})|0\rangle\,,\quad \tilde{H}N^*(\boldsymbol{p})|0\rangle = mN^*(p)|0\rangle\,, \\ \tilde{H}a^*(\boldsymbol{k})|0\rangle = \omega(\boldsymbol{k})a^*(\boldsymbol{k})|0\rangle\,.$$

Hence, it follows that the physical vacuum and physical one-particle states for the Hamiltonian (4) have the forms:

$$(11)\quad |0_{\mathrm{ph}}\rangle = U^{-1}|0\rangle = |0\rangle\,, \\ |V_{\mathrm{ph}}(\boldsymbol{p})\rangle = U^{-1}V^*(\boldsymbol{p})|0\rangle\,,\quad |N_{\mathrm{ph}}(\boldsymbol{p})\rangle = U^{-1}N^*(\boldsymbol{p})|0\rangle\,, \\ |\vartheta_{\mathrm{ph}}(\boldsymbol{k})\rangle = U^{-1}a^*(\boldsymbol{k})|0\rangle = a^*(\boldsymbol{k})|0\rangle\,.$$

The renormalization in the investigated model

In our model exact values for the renormalization constants Z_1, Z_2, Z_3 can be found [9]. Thus, one can perform the renormalization of the theory. In quantum electrodynamics we define the renormalization constants with the help of the formulae [10]:

$$(12)\quad S'_F(p) \underset{\hat{p}\to m}{=} Z_2S_F(p), \\ F'_\mu(p_1,p_2) \underset{\substack{\hat{p}_1\to m\\ \hat{p}_2\to m}}{=} Z_1^{-1}\gamma_\mu\,, \\ D'_F(k) \underset{k^2\to 0}{=} Z_3D_F(k)\,.$$

The analogous formulae in our theory have the forms:

$$G'_F(p) \underset{\substack{p_0\to m\\ \bar{p}=0}}{=} Z_2^{-1} G_F(p) = Z_2^{-1}\frac{i}{p_0 - m},$$

(13*)
$$\Gamma(p,q) \underset{\substack{p_0\to m\\ q_0\to m\\ \bar{p},\bar{q}=0}}{=} Z_1\sigma_1,$$

$$\Delta'_F(k) \underset{k^2\to\mu^2}{=} Z_3^{-1}\Delta_F(k) = Z_3^{-1}\frac{f^2(\omega)}{k^2-\mu^2}.$$

The functions G'_F, Δ'_F and $\Gamma(p,q)$ are given by Eqs. (11) and (12).

$$\langle 0|T(\psi(x_1)\psi^*(x_2))|0\rangle = \frac{1}{(2\pi)^4}\int d^4p e^{-ip(x_1-x_2)} G'_F(p),$$

$$\langle 0|T(\Phi(x_1)\Phi(x_2))|0\rangle = \frac{1}{(2\pi)^4}\int d^4k e^{-ik(x_1-x_2)} \Delta'_F(x),$$

(14)
$$\langle 0|T(\psi(x_1)\Phi(x)\psi^*(x_2))|0\rangle = \frac{1}{(2\pi)^8} g_0^{-1} Z_1 Z_2^{-1} Z_3^{-1/2}\cdot$$

$$\int d^4q_1 d^4q_2 e^{-iq_1(x_1-x)+iq_2(x_2-x)}\, G'_F(q_1)\Gamma(q,q_2) G'_F(q_2)\Delta'_F(q_2-q_1).$$

In our model one can compute exactly the propagators

$$\langle 0|T(\psi(x_1)\psi^*(x_2))|0\rangle,\ \langle 0|T(\psi(x_1)\Phi(x)\psi^*(x_2))|0\rangle,\ \langle 0|T(\Phi(x_1)\Phi(x_2))|0\rangle.$$

For example, we obtain for the fermion propagator:

(15)
$$\begin{aligned}\langle 0|T(\psi(x_1)\psi^*(x_2))|0\rangle &= \vartheta(t_1-t_2)\langle 0|\psi(x_1)\psi^*(x_2)|0\rangle\\ &= \vartheta(t_1-t_2)\langle 0|\psi(\boldsymbol{x}_1)e^{-iH(t_1-t_2)}\psi^*(\boldsymbol{x}_2)|0\rangle\\ &= \vartheta(t_1-t_2)\langle 0|U\psi(\boldsymbol{x}_1)U^{-1}e^{-i(\bar{H}_0+H_{\mathrm{II}})(t_1-t_2)}U\psi^*(\boldsymbol{x}_2)U^{-1}|0\rangle.\end{aligned}$$

Using the equations

$$[\bar{H}_0, H_{\mathrm{II}}] = 0, \qquad H_{\mathrm{II}}U\psi^*U^{-1}|0\rangle = 0$$

we obtain

(16)
$$\langle 0|T(\psi(x_1)\psi^*(x_2))|0\rangle =$$

$$= \vartheta(t_1-t_2)\left\langle 0\left|\exp\left(-ig_0\frac{1}{\omega^2}\pi(\boldsymbol{x}_1)\sigma_1\right)\psi(\boldsymbol{x}_1)e^{-iH_0(t_1-t_2)}\psi^*(\boldsymbol{x}_2)\exp\left(ig_0\frac{1}{\omega^2}\pi(x_2)\sigma_1\right)\right|0\right\rangle$$

$$= \vartheta(t_1-t_2)\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\exp\left[-\left(\frac{g_0}{g_c}\right)^2\right]e^{-im(t_1-t_2)}\exp\left[\frac{g_0^2}{2(2\pi)^3}\int d^3k\frac{f^2(\omega)}{\omega^2}e^{-i\omega(t_1-t_2)}\right],$$

*) In spite of the non-vanishing of the meson mass we can assume in this model the same conditions on Γ as in electrodynamics, because we do not take into account the change of the energy of the fermion.

where

$$g_c^{-2} = \frac{1}{2(2\pi)^3}\int d^3k \frac{f^2(\omega)}{\omega^2}\,. \tag{17}$$

We are interested only in that part of the propagator which contains the frequency m.

$$\langle 0\big|T(\psi(x_1)\psi^*(x_2))\big|0\rangle = \vartheta(t_1-t_2)\,e^{-(g_0/g_c)^2}\,\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\,e^{-im(t_1-t_2)}+\ldots \tag{18}$$

$$= e^{-(g_0/g_c)^2}\frac{i}{(2\pi)^4}\int d^4p e^{-ip_0(t_1-t_1)+i\boldsymbol{p}(\boldsymbol{x}_1-\boldsymbol{x}_2)}\frac{1}{p_0-m}+\ldots$$

Hence, it follows that

$$G'_F(p) \underset{\substack{p_0\to m\\ \boldsymbol{p}=0}}{=} e^{-(g_0/g_c)^2}\frac{i}{p_0-m}\,. \tag{19}$$

From Eq. (13) it follows that

$$Z_2 = e^{-(g_0/g_c)^2}\,. \tag{20}$$

In our model as in the Lee model

$$Z_3 = 1\,, \tag{21}$$

because there is no vacuum polarization. We compute in a similar manner the propagator $\langle 0\big|T(\psi(x_1)\Phi(x)\psi^*(x_2))\big|0\rangle$. Only the part containing the frequency m is necessary to compute the constant Z_1. We obtain (dropping other terms):

$$\langle 0\big|T(\psi(x_1)\Phi(x)\psi^*(x_2))\big|0\rangle \tag{22}$$

$$= -\frac{g_0}{(2\pi)^8}\sigma_1 e^{-(g_0/g_c)^2}\int d^4p d^4q\left[\frac{f(\vec{q}-\vec{p})}{\omega(\vec{q}-\vec{p})}\right]^2 e^{-ip(x_1-x)+iq(x_2-x)}\frac{1}{p_0-m}\cdot\frac{1}{p_0-m}+\ldots$$

Comparing the result with Eq. (14) we obtain immediately

$$Z_1 = e^{-(g_0/g_c)^2}\,. \tag{23}$$

Thus, in the model under consideration, the Ward identity takes place

$$Z_1 = Z_2\,. \tag{24}$$

The Lee model

The Lee model corresponding to the model considered above has been investigated by Källen and Pauli [2]. The Hamiltonian for this model arises from the Hamiltonian (1) by removing the terms describing the process $V+\vartheta\to N$, $N\to V+\vartheta$. As is well known, in such a model it

is necessary to renormalize the coupling constant. The connection between the unrenormalized and renormalized coupling constants is given by

$$g^2 = g_0^2 g_c^2 (g_c^2 + g_0^2)^{-1}. \tag{25}$$

This equation has the real solutions for g_0 only if $g^2 < g_c^2$.

$$H^L = H_0 + H_I^L,$$

$$H_0 = m_0 \int d^3p V^*(\boldsymbol{p}) V(\boldsymbol{p}) + m_0 \int d^3p N^*(\boldsymbol{p}) N(\boldsymbol{p}) + \int d^3k \omega(\boldsymbol{k}) a^*(\boldsymbol{k}) a(\boldsymbol{k}), \tag{26}$$

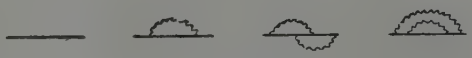
$$H_I^L = \frac{g_0}{(2\pi)^{3/2}} \int d^3p \int d^3k \frac{f(\omega)}{\sqrt{2\omega}} [V^*(\boldsymbol{p}) N(\boldsymbol{p}-\boldsymbol{k}) a(\boldsymbol{k}) + N^*(\boldsymbol{p}) V(\boldsymbol{p}+\boldsymbol{k}) a^*(\boldsymbol{k})].$$

In this model the renormalization constants have the values:

$$Z_1 = 1, \quad Z_2^{-1} = 1 + \frac{g_0^2}{g_c^2}, \quad Z_3 = 1. \tag{27}$$

Discussion

Let us take into consideration all diagrams of self energy of a fermion in the theory described by the Hamiltonian (1). The first of them have the forms:

(28) 

In our theory the simplest diagrams of the vertex part have the forms:

(29)

Thus, there exists an one to one correspondence between the diagrams of self energy and the vertex part. On account of this we have $Z_1 = Z_2$. In the Lee model described by the Hamiltonian (27) there appears only three of the diagrams (28) and (29).

(0)

We see from these considerations that the Lee model gives the zeroth approximation of the vertex part and the first approximation of the self energy. Thus, instead of the renormalization constants:

$$Z_1^{-1} = \exp(g_0/g_c)^2 = 1 + (g_0/g_c)^2 + \ldots$$
$$Z_2^{-1} = \exp(g_0/g_c)^2 = 1 + (g_0/g_c)^2 + \ldots \tag{30}$$

we have in the Lee model:

$$Z_1^{-1} = 1, \quad Z_2^{-1} = 1 + (g_0/g_c)^2 . \tag{31}$$

This lack of symmetry between the self energy and the vertex part causes the necessity of renormalization of the coupling constant. Thus, the difficulties in the Lee model are caused by removing half of the terms from Lee's interaction Hamiltonian. These difficulties do not exist in the usual local theory.

I wish to thank Professor L. Infeld for his kind interest. I also thank Professor W. Królikowski for valuable discussions.

INSTITUTE OF THEORETICAL PHYSICS, WARSAW UNIVERSITY
(INSTYTUT FIZYKI TEORETYCZNEJ UNIWERSYTETU WARSZAWSKIEGO)

REFERENCES

[1] T. D. Lee, Phys. Rev. **95** (1954), 1329.
[2] G. Källen and W. Pauli, Dan. Mat. Fys. Medd. **30** (1955), No 7.
[3] W. Heisenberg, Nucl. Phys. **4** (1957), 532.
[4] S. Tomonaga, Sci. Pap. I. P. C. R. **39** (1941), 247.
[5] G. Wentzel, *Quantum theory of fields*, New York 1949.
[6] L. van Hove, Physica **18** (1952), 145.
[7] — Bull. Acad. Roy. Belg., Cl. Sc. **27** (1951), 1055.
[8] O. W. Greenberg and S. S. Schweber, N. C. **8** (1958), 378.
[9] F. J. Dyson, Phys. Rev. **75** (1949), 1736.
[10] S. S. Schweber, H. A. Bethe and F. de Hoffmann, *Mesons and fields*, New York, 1955.
[11] J. Schwinger, Proc. Nat. Acad. Sci. U. S. A. **37** (1951), 452, 455.
[12] H. Umezawa, *Quantum field theory*, Amsterdam, 1956.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VII, No. 2, 1959

Electron Polarization in First Forbidden Unique Beta Transitions

by

B. KUCHOWICZ

Presented by L. INFELD on November 29, 1958

Formulae for the electron polarization in recoil experiments have been given so far only for allowed beta transitions [1]-[3]. No formulae of this kind have been hitherto computed for forbidden beta transitions, though the $e-\nu$ angular correlation formulae were computed long ago [4], [5]. Moreover, only longitudinal polarization without correlation has been computed for some theoretically simple cases: for $1\to 0$ transitions and scalar + tensor interaction in [6], for a $S+T+P$ interaction in [7] and for all pure interactions in the two-component theory in [8] *).

The electron can have transverse polarization components, as well as a longitudinal one. No formulae have been derived for these components in forbidden beta transitions. Thus, it seemed advisable to examine this question.

The aim of this paper is to investigate the electron polarization in the case of first forbidden unique beta transitions ($\Delta J = \pm 2$), where the treatment is essentially simple because under some plausible assumptions the nuclear matrix elements in the numerator and denominator of the polarization formulae cancel each other and a simple dependence on observables can be achieved. The same trace method of calculation with correct Coulomb functions as in [3] has been applied. The co-ordinate system is introduced in the following way: the electron momentum $\vec{p}$ is directed along the x-axis and the XOY-plane is fixed by the momenta

*) When the presented calculations were finished, the author received a copy of Phys. Rev. 111 (1958), 522 containing a paper by Berestetsky et al. which partly deals with the same problem. The results of the present paper coincide for a point nucleus with those obtained by Berestetsky et al. in all points but one. In fact, the formulae given in the paper of Berestetsky are less general since no difference is made between the pseudovector and tensor nuclear matrix elements.

$\vec{p}$ and $\vec{q}$ of the electron and neutrino. The z-axis has the direction of the vector-product $\vec{p} \times \vec{q}$.

In the subsequent formulae the energy E of the electron and the momenta are given in units of $m_e c^2$ and $m_e c$, respectively, where m_e denotes the rest mass of the electron, and c — the velocity of light. α is the fine structure constant; Z is the atomic number of the final nucleus, and $F_0(Z, E)$ is the Fermi function. The electron-neutrino correlation angle is ϑ. The unit vector $\vec{\zeta} = (n, t_y, t_z)$ gives the direction of the measurement of the electron polarization. The vector $\vec{\zeta}$ coincides with the spin vector of the particle in its rest system.

The squared transition matrix element from an initial nucleus state to a final nucleus state with the emission of an electron and an antineutrino has the form:

$$(1) \qquad |\langle H(\vec{p}, \vec{q}, Z, \vec{\zeta})\rangle|^2 = F_0(Z, E)[W + nN + t_y T_y \sin\vartheta + t_z T_z \sin\vartheta],$$

where the functions W, N, T_y and T_z are given below:

$$\begin{aligned}
(2) \quad W = {} & |C_T|^2(1+|\alpha_T|^2) \sum |B_{ij}^{\beta}|^2 \cdot T_1 + |C_A|^2(1+|\alpha_A|^2) \sum |B_{ij}|^2 \cdot A_1 \\
& + 2\,\mathrm{Re}\Big[C_T C_A^*(1+\alpha_T \alpha_A^*)\Big]\Big\{\mathrm{Re}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_1 + \mathrm{Im}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_2\Big\} \\
& + 2\,\mathrm{Im}\Big[C_T C_A^*(1+\alpha_T \alpha_A^*)\Big]\Big\{\mathrm{Re}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_2 - \mathrm{Im}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_1\Big\}
\end{aligned}$$

$$\begin{aligned}
N = {} & |C_T|^2(\alpha_T + \alpha_A^*) \sum |B_{ij}^{\beta}|^2 \cdot T_2 + |C_A|^2(\alpha_A + \alpha_A^*) \sum |B_{ij}|^2 \cdot A_2 \\
& + 2\,\mathrm{Re}\Big[C_T C_A^*(\alpha_T + \alpha_A^*)\Big]\Big\{\mathrm{Re}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_3 + \mathrm{Im}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_4\Big\} \\
& + 2\,\mathrm{Im}\Big[C_T C_A^*(\alpha_T + \alpha_A^*)\Big]\Big\{\mathrm{Re}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_4 - \mathrm{Im}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_3\Big\}
\end{aligned}$$

$$\begin{aligned}
T_y = {} & |C_T|^2(\alpha_T + \alpha_T^*) \sum |B_{ij}^{\beta}|^2 \cdot T_3 + |C_A|^2(\alpha_A + \alpha_A^*) \sum |B_{ij}|^2 \cdot A_3 \\
& + 2\,\mathrm{Re}\Big[C_T C_A^*(\alpha_T + \alpha_A^*)\Big]\Big\{\mathrm{Re}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_5 + \mathrm{Im}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_6\Big\} \\
& + 2\,\mathrm{Im}\Big[C_T C_A^*(\alpha_T + \alpha_A^*)\Big]\Big\{\mathrm{Re}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_6 - \mathrm{Im}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_5\Big\}
\end{aligned}$$

$$\begin{aligned}
T_z = {} & |C_T|^2(1+|\alpha_T|^2) \sum |B_{ij}^{\beta}|^2 \cdot T_4 + |C_A|^2(1+|\alpha_A|^2) \sum |B_{ij}|^2 \cdot A_4 \\
& + 2\,\mathrm{Re}\Big[C_T C_A^*(1+\alpha_T \alpha_A^*)\Big]\Big\{\mathrm{Re}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_7 + \mathrm{Im}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_8\Big\} \\
& + 2\,\mathrm{Im}\Big[C_T C_A^*(1+\alpha_T \alpha_A^*)\Big]\Big\{\mathrm{Re}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_8 - \mathrm{Im}\Big[\sum (B_{ij}^{\beta} B_{ij}^*)\Big] \cdot I_7\Big\}.
\end{aligned}$$

Here $C_T \alpha_T$ and $C_A \alpha_A$ are the coupling constants of the space parity non-conserving interactions in the notation of [3]. B_{ij}^{β} is the tensor- and

B_{ij} – the pseudovector nuclear matrix element in standard notation. The coefficients T_i, A_i and I_i are listed below:

$$T_1 = {}_{1}{}^{1}{}_{2}\, q^2 L_0 + {}_{4}^{3} L_1 + [{}_{6}{}^{1}{}_{0}\, q^2 L_0' + \tfrac{1}{2} q K_{c1,2}^{+} + {}_{2}{}^{3}{}_{0}\, L_1'] \cos\vartheta - {}_{2}{}^{3}{}_{0}\, q \hat{K}_{c1,-2}^{+} (\tfrac{1}{3} - \cos^2\vartheta)$$

$$T_2 = {}_{1}{}^{1}{}_{2}\, q^2 L_0' + {}_{4}^{3} L_1' + [{}_{6}{}^{1}{}_{0}\, q^2 L_0 + \tfrac{1}{2} q \hat{K}_{c1,-2}^{+} + {}_{2}{}^{3}{}_{0}\, L_1] \cos\vartheta - {}_{2}{}^{3}{}_{0}\, q K_{c1,2}^{+} (\tfrac{1}{3} - \cos^2\vartheta)$$

$$T_3 = {}_{6}{}^{1}{}_{0}\, q^2 P_0 - \tfrac{1}{4} q \hat{K}_{c1,-2}^{-} + {}_{1}{}^{3}{}_{0}\, P_1 - {}_{2}{}^{3}{}_{0}\, q \cos\vartheta \cdot K_{c1,2}^{-}$$

$$T_4 = {}_{6}{}^{1}{}_{0}\, q^2 P_0' + \tfrac{1}{4} q K_{s1,2}^{-} + {}_{1}{}^{3}{}_{0}\, P_1' - {}_{2}{}^{3}{}_{0}\, q \cos\vartheta \cdot \hat{K}_{s1,-2}^{-}$$

$$A_1 = {}_{1}{}^{1}{}_{2}\, q^2 L_0 + {}_{4}^{3} L_1 + [- {}_{6}{}^{1}{}_{0}\, q^2 L_0' + \tfrac{1}{2} q K_{c1,2}^{+} - {}_{2}{}^{3}{}_{0}\, L_1'] \cos\vartheta + {}_{2}{}^{3}{}_{0}\, q \hat{K}_{c1,-2}^{+} (\tfrac{1}{3} - \cos^2\vartheta)$$

$$A_2 = - {}_{1}{}^{1}{}_{2}\, q^2 L_0' - {}_{4}^{3} L_1' + [{}_{6}{}^{1}{}_{0}\, q^2 L_0 - \tfrac{1}{2} q \hat{K}_{c1,-2}^{+} + {}_{2}{}^{3}{}_{0}\, L_1] \cos\vartheta - {}_{2}{}^{3}{}_{0}\, q K_{c1,2}^{+} (\tfrac{1}{3} - \cos^2\vartheta)$$

$$A_3 = {}_{6}{}^{1}{}_{0}\, q^2 P_0 + \tfrac{1}{4} q \hat{K}_{c1,-2}^{-} + {}_{1}{}^{3}{}_{0}\, P_1 - {}_{2}{}^{3}{}_{0}\, q \cos\vartheta \cdot K_{c1,2}^{-}$$

$$A_4 = - {}_{6}{}^{1}{}_{0}\, q^2 P_0' + \tfrac{1}{4} q K_{s1,2}^{-} - {}_{1}{}^{3}{}_{0}\, P_1 + {}_{2}{}^{3}{}_{0}\, q \cos\vartheta \cdot \hat{K}_{s1,-2}^{-}$$

$$I_1 = {}_{1}{}^{1}{}_{2}\, q^2 P_0 + {}_{4}^{3} P_1 - \tfrac{1}{2} q \cos\vartheta \cdot K_{c1,2}^{-}$$

$$I_2 = -({}_{6}{}^{1}{}_{0}\, q^2 P_0' + {}_{2}{}^{3}{}_{0}\, P_1') \cos\vartheta + {}_{2}{}^{3}{}_{0}\, q (\tfrac{1}{3} - \cos^2\vartheta) \hat{K}_{s1,-2}^{-}$$

$$I_3 = ({}_{6}{}^{1}{}_{0}\, q^2 P_0 + {}_{2}{}^{3}{}_{0}\, P_1) \cos\vartheta + {}_{2}{}^{3}{}_{0}\, q (\tfrac{1}{3} - \cos^2\vartheta) K_{c1,2}^{-}$$

$$I_4 = -({}_{1}{}^{1}{}_{2}\, q^2 P_0' + {}_{4}^{3} P_1' + \tfrac{1}{2} q \cos\vartheta \cdot \hat{K}_{s1,-2}^{-})$$

$$I_5 = {}_{6}{}^{1}{}_{0}\, q^2 L_0 + {}_{1}{}^{3}{}_{0}\, L_1 + {}_{2}{}^{3}{}_{0}\, q \cos\vartheta \cdot K_{c1,2}^{+}$$

$$I_6 = - \tfrac{1}{4} q \hat{K}_{s1,-2}^{+}$$

$$I_7 = - \tfrac{1}{4} q K_{s1,2}^{+}$$

$$I_8 = - {}_{6}{}^{1}{}_{0}\, q^2 L_0' - {}_{1}{}^{3}{}_{0}\, L_1' - {}_{2}{}^{3}{}_{0}\, q \cos\vartheta \cdot \hat{K}_{c1,-2}^{+} .$$

The functions L_i and P^i are the usually used beta decay functions defined e. g. in [9], and the functions L_i' have been introduced by Curtis and Lewis [6]. There is need for defining some new combinations of Coulomb radial wave functions. These new functions are the P_i' and K-functions. They are defined in the Appendix.

The longitudinal polarization P_x of the electron and the transverse polarization components P_y and P_z are given by

$$P_x = \frac{N}{W}, \qquad P_y = \frac{T_y}{W} \sin\vartheta, \qquad P_z = \frac{T_z}{W} \sin\vartheta .$$

The formulae have been given so far only for electron decay. The formulae for positron decay may be obtained from the preceding ones in the usual way by changing $Z \to -Z$, $a_i \to - a_i^*$ and $C_S \to - C_S^*$, $C_A \to - C_A^*$, $C_P \to - C_P^*$, $C_T \to C_T^*$, $C_V \to C_V^*$.

When the limiting values of the beta decay functions $\left(\text{for } \frac{\alpha Z}{p} \to 0 \text{ and } p\varrho \to 0\right)$ are used, the four functions W, N, T_y and T_z from the formulae (1) and (4) have the following asymptotic form:

$$
\begin{aligned}
(5)\quad W = {} & |C_T|^2(1+|a_T|^2)\sum |B_{ij}^{\beta}|^2\cdot\left\{\frac{1}{12}(p^2+q^2)+\left[\frac{1}{60}(p^2+q^2)+\frac{1}{6}qE\right]\frac{p}{E}\cos\vartheta\right. \\
& \left.-\frac{1}{20}\frac{p^2q}{E}\left(\frac{1}{3}-\cos^2\vartheta\right)\right\}+|C_A|^2(1+|a_A|^2)\sum |B_{ij}|^2\cdot\left\{\frac{1}{12}(p^2+q^2)\right. \\
& \left.+\left[-\frac{1}{60}(p^2+q^2)+\frac{1}{6}qE\right]\frac{p}{E}\cos\vartheta+\frac{1}{20}\frac{p^2q}{E}\left(\frac{1}{3}-\cos^2\vartheta\right)\right\} \\
& +2\,\mathrm{Re}[C_T C_A^*(1+a_T a_A^*)]\left\{\pm\mathrm{Re}\left[\sum (B_{ij}^{\beta}B_{ij}^*)\right]\cdot\frac{1}{E}\left[\frac{1}{12}(p^2+q^2)\right.\right. \\
& \left.\left.+\frac{1}{6}pq\cos\vartheta\right]+\mathrm{Im}\left[\sum (B_{ij}^{\beta}B_{ij}^*)\right]\cdot\frac{\alpha Zp}{E}\left[\frac{1}{60}(q^2+\tfrac{1}{2}p^2)\cos\vartheta\right.\right. \\
& \left.\left.-\frac{3}{80}q\left(\frac{1}{3}-\cos^2\vartheta\right)\right]\right\}+2\,\mathrm{Im}\left[C_T C_A^*(1+a_T a_A^*)\right]\left\{\pm\mathrm{Re}\left[\sum (B_{ij}^{\beta}B_{ij}^*)\right]\right. \\
& \left.\times\frac{\alpha Zp}{E}\left[\frac{1}{60}(q^2+\tfrac{1}{2}p^2)\cos\vartheta-\frac{3}{80}q\left(\frac{1}{3}-\cos^2\vartheta\right)\right]\right. \\
& \left.-\mathrm{Im}\left[\sum (B_{ij}^{\beta}B_{ij}^*)\right]\frac{1}{E}\left[\frac{1}{12}(p^2+q^2)+\frac{1}{6}pq\cos\vartheta\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
N = {} & \pm|C_T|^2(a_T+a_T^*)\sum |B_{ij}^{\beta}|^2\cdot\left\{\frac{1}{12}(p^2+q^2)\frac{p}{E}+\left[\frac{1}{60}(p^2+q^2)+\frac{1}{6}q\frac{p^2}{E}\right]\cos\vartheta\right. \\
& \left.-\frac{1}{20}qp\left(\frac{1}{3}-\cos^2\vartheta\right)\right\}\pm|C_A|^2(a_A+a_A^*)\sum |B_{ij}|^2\left\{-\frac{1}{12}(p^2+q^2)\frac{p}{E}\right. \\
& \left.+\left(\frac{1}{60}(p^2+q^2)-\frac{1}{6}q\frac{p^2}{E}\right]\cos\vartheta-\frac{1}{20}qp\left(\frac{1}{3}-\cos^2\vartheta\right)\right\} \\
& +2\,\mathrm{Re}[C_T C_A^*(a_T+a_A^*)]\left\{\mathrm{Re}\left[\sum (B_{ij}^{\beta}B_{ij}^*)\right]\cdot\frac{1}{E}\left[\frac{1}{60}(p^2+q^2)\cos\vartheta\right.\right. \\
& \left.\left.-\frac{1}{20}pq\left(\frac{1}{3}-\cos^2\vartheta\right)\right]\pm\mathrm{Im}\left[\sum (B_{ij}^{\beta}B_{ij}^*)\right]\cdot\frac{\alpha Zp}{E}\left[\frac{1}{12}(q^2+\tfrac{1}{2}p^2)\right.\right. \\
& \left.\left.+\frac{1}{8}q\cos\vartheta\right]\right\}+2\,\mathrm{Im}[C_T C_A^*(a_T+a_A^*)]\left\{\mathrm{Re}\left[\sum (B_{ij}^{\beta}B_{ij}^*)\right]\right. \\
& \left.\times\frac{\alpha Zp}{E}\left[\frac{1}{12}(q^2+\tfrac{1}{2}p^2)+\frac{1}{8}q\cos\vartheta\right]\mp\mathrm{Im}\left[\sum (B_{ij}^{\beta}B_{ij}^*)\right]\frac{1}{E}\left[\frac{1}{60}(p^2+q^2)\cos\vartheta\right.\right. \\
& \left.\left.-\frac{1}{20}pq\left(\frac{1}{3}-\cos^2\vartheta\right)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
T_y = & \pm |C_T|^2(a_T + a_T^*) \sum |B_{ij}^\beta|^2 \cdot \frac{1}{E}\left[\frac{1}{60}(q^2+2p^2) + \frac{1}{20}pq\cos\vartheta\right] \\
& \pm |C_A|^2(a_A + a_A^*) \sum |B_{ij}|^2 \cdot \frac{1}{E}\left[\frac{1}{60}(q^2+2p^2) + \frac{1}{20}pq\cos\vartheta\right] \\
& + 2\,\mathrm{Re}[C_T C_A^*(a_T + a_A^*)]\left\{\mathrm{Re}\left[\sum (B_{ij}^\beta B_{ij}^*)\right]\cdot\left[\frac{1}{60}(q^2+2p^2) + \frac{1}{20}pq\cos\vartheta\right]\right. \\
& \left. \mp \mathrm{Im}\left[\sum (B_{ij}^\beta B_{ij}^*)\right]\cdot\frac{\alpha Z}{16}pq\right\} + \mathrm{Im}[C_T C_A^*(a_T + a_A^*)] \\
& \times \left\{-\mathrm{Re}\left[\sum (B_{ij}^\beta B_{ij}^*)\right]\cdot\frac{\alpha Z}{16}pq \mp \mathrm{Im}\left[\sum (B_{ij}^\beta B_{ij}^*)\right]\right. \\
& \left.\times\left[\frac{1}{60}(q^2+2p^2) + \frac{1}{20}pq\cos\vartheta\right]\right\} \\
T_z = & \pm |C_T|^2(1+|a_T|^2) \sum |B_{ij}^\beta|^2 \cdot \alpha Z\left[-\frac{1}{60E}(p^2+q^2) - \frac{1}{12}q + \frac{3}{80}\frac{qp}{E}\cos\vartheta\right] \\
& \pm |C_A|^2(1+|a_A|^2) \sum |B_{ij}|^2 \alpha Z\left[\frac{1}{60E}(p^2+q^2) - \frac{1}{12}q - \frac{3}{80}\frac{qp}{E}\cos\vartheta\right] \\
& + 2\,\mathrm{Re}[C_T C_A^*(1 + a_T a_A^*)]\left\{-\mathrm{Re}\left[\sum (B_{ij}^\beta B_{ij}^*)\right]\frac{\alpha Z}{4}q\left(\frac{p^2}{4E} + \frac{1}{3E}\right)\right. \\
& \left. \mp \mathrm{Im}\left[\sum (B_{ij}^\beta B_{ij}^*)\right]\frac{p}{E}\left[\frac{1}{60}(q^2+2p^2) + \frac{1}{20}pq\cos\vartheta\right]\right\} \\
& + 2\,\mathrm{Im}[C_T C_A^*(1 + a_T a_A^*)]\left\{-\mathrm{Re}\left[\sum (B_{ij}^\beta B_{ij}^*)\right]\cdot\frac{p}{E}\left[\frac{1}{60}(q^2+2p^2)\right.\right. \\
& \left.\left. + \frac{1}{20}pq\cos\vartheta\right] \pm \mathrm{Im}\left[\sum (B_{ij}^\beta B_{ij}^*)\right]\frac{\alpha Z}{4}q\left(\frac{p^2}{4E} + \frac{1}{3E}\right)\right\}.
\end{aligned}
$$

In formulae (5), the upper signs refer to electron decay, the lower ones — to positron decay. These formulae are approximations, valid for $\alpha Z\, p \ll 1$ and $p\varrho \ll 1$ (where ϱ is the nuclear radius). This is no strong restriction on the electron momenta, because in our unit system we should have $\alpha Z \ll p \ll 10^3 \cdot A^{-1/3}$, and this condition is always practically fulfilled for light nuclei and electrons of kinetic energy greater than ~20 keV.

The interference terms in the formulae (2) and (5) have been grouped in such a way as to make it easy to cancel some of them when special assumptions are made. The imaginary part of the product of nuclear matrix elements: $\mathrm{Im}\left|\sum (B_{ij}^\beta B_{ij}^*)\right|$ will be equal to zero if one of the following assumptions is made:

1) time reversal invariance is valid for the nuclear forces, or

2) no difference is made between the nuclear matrix elements for the pseudovector and tensor couplings (i. e. we put $\beta \simeq 1$).

While the assumption that the imaginary part of the product of nuclear matrix elements is equal to zero has thus been partially justified, the imaginary part of the product of coupling constants cannot be can-

celled because it is the aim of the present investigations to establish, whether time reversal invariance is violated in β decay. Thus, it is of great importance for the experiment to give the possible arrangements in which the terms violating time reversal invariance can be detected. Such arrangements for allowed transitions have been discussed in a preceding paper [3]. It is possible, in principle, to make similar investigations in the present case, but the formulae are more complicated, and it will be difficult to arrive at a definite conclusion. One test, however, seems to give a definite answer. This is the measurement of the transverse z-component of the electron polarization together with the $e-\nu$ correlation angle.

If only the real part of the products of nuclear matrix elements is retained, a definite conclusion from the formulae is as follows: the z-polarization is made up of two terms, the one proportional to αZ (Coulomb effect) and the second due to the violation of time reversal invariance. The z-polarization due to the Coulomb effect can be estimated for any transition given. If time reversal invariance does not hold and the experiment is performed with high energetic electrons, the second term can dominate over the Coulomb term. If the experimental value of the z-polarization exceeds the theoretically estimated maximum value due to the Coulomb effect, this will be a positive proof of the violation of time reversal invariance. So far, we have considered only the terms in the numerator of the expression for P_z, but even when the terms in the denominator are taken into account our idea of estimation remains the same. Such estimation has been done in a preliminary note of the author [11].

Besides application to the search for time reversal invariance, the formulae given in this paper can be used for other investigations, which do not seem impossible in view of the development of experimental technique of $e-\nu$ angular correlation. If the correlation-polarization measurements are performed for several elements, the interaction type of the β decay may be completely decided, e. g. in the first forbidden unique transitions it can be decided, whether the type of the interaction is T or A.

The author wishes to express his cordial appreciation to Dr J. Werle for his kind suggestions and discussions.

Appendix

The following new combinations of Coulomb radial wave functions are introduced:

$$P_0' = [2p^2F_0(Z, E)]^{-1} \cdot 2f_1g_{-1}\sin(\Delta_1 - \Delta_{-1}) \to -\frac{\alpha Z}{E}$$

$$P_1' = [2p^2F_0(Z, E)]^{-1} \cdot \varrho^{-2} \cdot 2f_2g_{-2}\sin(\Delta_2 - \Delta_{-2}) \to -\frac{\alpha Z}{2} \cdot \frac{p^2}{9E}$$

$$K^{+}_{c1,2} = [2p^2F_0(Z,E)\varrho]^{-1}\cdot[f_1f_2\cos(\Delta_1-\Delta_2)+g_{-1}g_{-2}\cos(\Delta_{-1}-\Delta_{-2})]\to\frac{p}{3}$$

$$K^{-}_{c1,2} = [2p^2F_0(Z,E)\varrho]^{-1}\cdot[f_1f_2\cos(\Delta_1-\Delta_2)-g_{-1}g_{-2}\cos(\Delta_{-1}-\Delta_{-2})]\to-\frac{p}{3E}$$

$$K^{+}_{s1,2} = [2p^2F_0(Z,E)\varrho]^{-1}\cdot[f_1f_2\sin(\Delta_1-\Delta_2)+g_{-1}g_{-1}\sin(\Delta_{-1}-\Delta_{-2})]\to$$
$$\to\alpha Z\left(\frac{1}{3E}+\frac{p^2}{4E}\right)$$

$$K^{-}_{s1,2} = [2p^2F_0(Z,E)\varrho]^{-1}\cdot[f_1f_2\sin(\Delta_1-\Delta_2)-g_{-1}g_{-2}\sin(\Delta_{-1}-\Delta_{-2})]\to-\tfrac{1}{3}\alpha Z$$

$$\hat{K}^{+}_{c1,-2} = [2p^2F_0(Z,E)\varrho]^{-1}\cdot[f_1g_{-2}\cos(\Delta_1-\Delta_{-2})+g_{-1}f_2\cos(\Delta_{-1}-\Delta_2)]\to\frac{p^2}{3E}$$

$$\hat{K}^{-}_{c1,-2} = [2p^2F_0(Z,E)\varrho]^{-1}\cdot[f_1g_{-2}\cos(\Delta_1-\Delta_{-2})-g_{-1}f_2\cos(\Delta_{-2}-\Delta_2)]\to0$$

$$\hat{K}^{+}_{s1,-2} = [2p^2F_0(Z,E)\varrho]^{-1}\cdot[f_1g_{-2}\sin(\Delta_1-\Delta_{-2})+g_{-1}f_2\sin(\Delta_{-1}-\Delta_2)]\to\tfrac{1}{4}\alpha Zp$$

$$\hat{K}^{-}_{s1,-2} = [2p^2F_0(Z,E)\varrho]^{-1}\cdot[f_1g_{-2}\sin(\Delta_1-\Delta_{-2})-g_{-1}f_2\sin(\Delta_{-1}-\Delta_2)]\to-\tfrac{1}{4}\alpha Z\frac{p}{E}.$$

The arrows indicate the limiting values of these functions for $\alpha Z/p \ll 1$ and $p\varrho \ll 1$. The functions f_k and g_k are the radial wave functions given by Rose, normalized to one particle per unit sphere, and the Coulomb phase shifts Δ_k are defined at in [10].

Added in proof: The same calculation method has been used by the author in his further calculations of first forbidden transitions. Formulae have been obtained for the $e-\nu$ correlation and electron polarization in general first for bidden transitions, and for the $A-V$ couplings, which are now of great interest. The results are being published.

INSTITUTE FOR NUCLEAR RESEARCH, WARSAW
(INSTYTUT BADAŃ JĄDROWYCH W WARSZAWIE)

REFERENCES

[1] J. D. Jackson, S. B. Treiman, H. W. Wyld, Phys. Rev. **106** (1957), 517.
[2] — Nucl. Phys. **4** (1957), 206.
[3] B. Kuchowicz, Inst. Nucl. Research (Warsaw) Report 25/VII; Bull. Acad. Pol. Sci. Sér. sci. math. astr. et phys., **6** (1958), 395.
[4] E. Greuling, M. L. Meeks, Phys. Rev. **82** (1951), 531.
[5] M. Morita, Progr. Theor. Phys. **9** (1953), 345.
[6] R. B. Curtis, R. R. Lewis, Phys. Rev. **107** (1957), 543.
[7] K. Adler, B. Stech, A. Winther, Phys. Rev. **107** (1957), 328.
[8] A. Deloff, Bull. Acad. Pol. Sci. sér. sci. math., astr. et phys., **6** (1958), 57.
[9] K. Siegbahn, *Beta- and Gamma-Ray Spectroscopy*, North Holland Publ. Comp., Amsterdam, 1955, 884.
[10] M. Rose, L. Biedenharn, G. Arfken, Phys. Rev. **85** (1952), 5.
[11] B. Kuchowicz, Zs. f. Naturforschg, **13a** (1958), 999.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 2, 1959

On the Equivalence of a Certain Class of Non-Local and Higher Order Field Theories

by

K. L. NAGY and J. RZEWUSKI

Presented by L. INFELD on December 5, 1958

In a previous paper one of us (J. R.) considered the reduction of the problem of quantization of a certain class of non-local field theories to the problem of canonical quantization of fields possessing higher order Lagrangians [1]. In this paper, however, the connection of the two Hilbert spaces and especially the connection of the two S-matrices have not been investigated.

The other author (K. L. N.) has shown explicitly [2] that Bogolubov's method [3], proposed for the elimination of the non-physical consequences of the indefinite metric, is equivalent to an ordinary but non-local theory.

As a synthesis of these two papers we now show that a certain class of non-local field theories is equivalent to field theories with higher order Lagrangians, provided that in the Hilbert space of the latter are applied Bogolubov's prescriptions [2] of a certain kind. These prescriptions also eliminate all the well known difficulties of a higher order theory. Since relativistic higher order differential equations do not violate the principle of causality, thus just the Bogolubov's prescription brings in the non-causalities typical for a non-local theory (cf. [3]).

For the sake of simplicity we shall consider here the non-local interaction of a neutral scalar field φ with a complex scalar field ψ. The field equations are supposed to be

$$
\begin{aligned}
(\Box - m^2)\varphi(x) &= g\int F(x-x')\psi^*(x')\psi(x')\,dx', \\
(\Box - \varkappa^2)\psi(x) &= g\int dx'\varphi(x')F(x'-x)\psi(x)\,.
\end{aligned}
\tag{1}
$$

(The method which follows can also be applied to more general form-functions [1], [4]). By means of the Yang-Feldman formalism the in-

and out-going fields can be defined satisfying the commutation relations

$$\left[\overset{\text{in}}{\underset{\text{out}}{\varphi}}(x), \overset{\text{in}}{\underset{\text{out}}{\varphi}}(x')\right] = i\Delta_m(x-x'),$$
$$\left[\overset{\text{in}}{\underset{\text{out}}{\psi}}, \overset{\text{in}}{\underset{\text{out}}{\psi^*}}(x')\right] = i\Delta_\varkappa(x-x'), \tag{2}$$

and the (unitary) S-matrix of this problem, containing the ingoing fields only, can be determined [5], [6]. This matrix will further be denoted as S_{nl}.

We assume that there exists an operator $F^{-2}(\Box)$ of the form

$$F^{-2}(\Box) = \prod_i \left(1-\frac{\Box}{\mu_i^2}\right), \tag{3}$$

such that

$$F^{-1}(\Box)F(x) = \delta(x),$$

where $\mu_i \neq \mu_j \neq m \neq \varkappa$, μ_i real, and proceed to show the equivalence.

Applying (3) to the field equations (1) we obtain (cf. [1])

$$\prod_i \left(1-\frac{\Box}{\mu_i^2}\right)(\Box - m^2)\tilde{\varphi}(x) = g\psi^*(x)\psi(x), \tag{4}$$
$$(\Box - \varkappa^2)\psi(x) = g\tilde{\varphi}(x)\psi(x),$$

where

$$\tilde{\varphi}(x) = \int dx'\varphi(x')F(x'-x).$$

It may be noted that, for a solution $\varphi^0(x)$ of the unperturbed equation $(\Box - m^2)\varphi^0 = 0$, $\tilde{\varphi}^0 = F(-m^2)\varphi^0$, where $F(-m^2)$ is the Fourier transform of $F(x)$ taken at the point $k_\mu^2 = -m^2$. This constant factor may always be normalized to unity so that $\tilde{\varphi}^0 = \varphi^0$ (cf. also [1]).

Equations of type (4) possess a larger manifold of solutions than Eq. (1). Denoting the general solutions of (4) by Φ and Ψ we have

$$\prod_i \left(1-\frac{\Box}{\mu_i^2}\right)(\Box - m^2)\Phi(x) = g\Psi^*(x)\Psi(x), \tag{4'}$$
$$(\Box - \varkappa^2)\Psi(x) = g\Phi(x)\Psi(x).$$

Eqs. (4') can be treated in the conventional manner. As shown by Pais and Uhlenbeck [7], a field Φ of the type (4') forms an assembly of single mass root fields:

$$\Phi^{(0)} = F^{-2}\Phi,$$
$$\Phi^{(i)} = -\prod_j{}' \left(1-\frac{\Box}{\mu_j^2}\right)\frac{\Box - m^2}{\mu_i^2}\Phi, \tag{5}$$

satisfying the following field equations and commutation relations

$$(\Box - m^2)\Phi^{(0)}(x) = g\Psi^*(x)\Psi(x)\,, \tag{6}$$

$$(\Box - \mu_i^2)\Phi^{(i)}(x) = g\Psi^*(x)\Psi(x)\,,$$

$$[\Phi^{(i)}(x), \Phi^{(j)}(x')] = i\delta_{ij}\frac{1}{\xi_i}\Delta_{\mu_i}(x-x')\,,$$

where

$$\xi_0 = 1\,, \qquad \xi_i = [\mu_i^2 F^{-2'}(\mu_i^2)]^{-1}\,.$$

The fields possessing sign $\xi = -1$ can be quantized by means of indefinite metric [8], [9].

Now we can easily go over to the interaction picture. We assume that the Heisenberg and the interaction pictures coincide at $-\infty$. In this case the interaction picture operators and the ingoing field operators are the same

$$\Phi_i = \underset{\text{in}}{\Phi}\,, \qquad \Psi_i = \underset{\text{in}}{\Psi}\,.$$

In the interaction picture we can easily write down the S-matrix explicitly in the form

$$S = P\exp\left\{-ig\int \Psi_i^*\Psi_i\Phi_i dx\right\}\,, \tag{7}$$

where

$$\Phi_i(x) = \sum_k \xi_k \Phi_i^{(k)}(x)\,.$$

For the elimination of non-physical consequences of the fields $\Phi^{(i)}$ we apply Bogolubov's prescriptions. We assume that only the particles of the $\Phi^{(0)}$ and Ψ fields posses a physical meaning. State vectors describing only these types of particles (and vacuum) will be denoted by $|\rangle_p$. The projection operator projecting the state vector into $|\rangle_p$ will be denoted by P_p:

$$P_p|\rangle = |\rangle_p\,, \quad 1 - P_p = P_n\,, \quad P_n|\rangle = |\rangle_n\,.$$

We assume that the non-physical parts of the state vectors are determined by the physical parts

$$|\rangle = |\rangle_p + N|\rangle_p\,, \qquad P_n N P_p = N\,.$$

Thus, the physical S-matrix defined by

$$|\infty\rangle_p = \widetilde{S}|-\infty\rangle_p$$

is

$$\widetilde{S} = P_p S(P_p + N)\,. \tag{8}$$

This method eliminates the non-physical consequences of the theory (e. g. indefiniteness of the metric or energy).

The proof of the equivalence is now quite simple. We identify the operators

$$\underset{\text{in}}{\varphi} = \underset{\text{in}}{\Phi^{(0)}} = \underset{\text{in}}{\Phi_i^{(0)}}, \qquad \underset{\text{in}}{\psi} = \underset{\text{in}}{\Psi} = \underset{\text{in}}{\Psi_i} \tag{9}$$

(note that $\underset{\text{in}}{\varphi} = \underset{\text{in}}{\tilde{\varphi}}$) satisfying the same field equations and commutation relations, and choose

$$N = (P_p S P_n)^{-1} (S_{nl} - P_p S P_p),$$

where

$$P_p S P_n (P_p S P_n)^{-1} = P_p, \qquad (P_p S P_n)^{-1} P_p S P_n = P_n.$$

In this case

$$\tilde{S} = S_{nl}. \tag{10}$$

The physical S-matrix defined above does not agree in general with Bogolubov's one. Thus, a procedure of this type might just serve as a guiding principle for the choice of a physically reasonable N from the great variety of possible different forms.

One of the authors (K. L. N.) enjoyed the hospitality in Poland of the Institute for Nuclear Research of the Polish Academy of Sciences and is greatly indebted for it.

INSTITUTE OF THEORETICAL PHYSICS, EÖTVÖS LORÁND UNIVERSITY, BUDAPEST
INSTITUTE OF THEORETICAL PHYSIC, WROCŁAW UNIVERSITY
(INSTYTUT FIZYKI TEORETYCZNEJ UNIWERSYTETU WROCŁAWSKIEGO)
INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT FIZYKI PAN, ODDZIAŁ WROCŁAWSKI)

REFERENCES

[1] J. Rzewuski, Acta Phys. Pol. **12** (1953), 100.
[2] K. L. Nagy, Nuovo Cimento (in press).
[3] N. N. Bogolubov, B. V. Medvedev and M. K. Polivanov, preprint, J. I. N. R. 1958; CERN Conference Reports, **129** (1958).
[4] J. Rzewuski, Acta Phys. Pol. **12** (1953), 14.
[5] J. Rayski, Acta Phys. Pol. **11** (1951), 25.
[6] P. Kristensen and C. Moller, Danske Vidensk. Selskab, Mat. fys. Meddelelser, **27** (1952), No. 7.
[7] A. Pais and G. E. Uhlenbeck, Phys. Rev. **79** (1950), 145.
[8] P. T. Matthews, Proc. Cambr. Phil. Soc. **45** (1949), 441.
[9] R. J. N. Phililps, Nuovo Cimento, **1** (1955), 823.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 2, 1959

On the Second Neutral Pion

by

W. KRÓLIKOWSKI

Presented by W. RUBINOWICZ on December 5, 1958

A number of authors [1]-[6] considered recently a possibility of the existence of a second neutral pion π_0^0 being a charge singulet, whereas the usual pion π^0 belongs to the charge triplet π^+, π^0, π^-. It was also pointed out [7] that this hypothetical particle might play a role in weak interactions leading to mesonic decays of hyperons. Namely, if the field π_0^0 existed, it would be possible to group the four fields π^+, π^0, π^-, π_0^0, not only in one isotriplet ($T = 1$, $S = 0$) and one isosingulet ($T = 0$, $S = 0$), but also, alternatively, in two doublets

$$(1) \qquad \Pi_a = \begin{cases} \pi^+ \\ \dfrac{1}{\sqrt{2}}(\pi^0 + i\pi_0^0) \end{cases} \quad \text{and} \quad \Pi_a^* = \begin{cases} \pi^- \\ \dfrac{1}{\sqrt{2}}(\pi^0 - i\pi_0^0) \end{cases}$$

having "apparent" isospin $T = \frac{1}{2}$ and "apparent" strangeness $S = \mp 1$ The asterisk denotes here the hermitian conjugation. Assuming that the grouping (1) would be proper for weak interactions leading to decays of hyperons (whereas the usual one would remain proper for strong interactions), we may interpret the Gell-Mann rule $|\Delta T_3| = \frac{1}{2}$ or $|\Delta S| = 1$ for "true" isospin and strangeness as a conservation law $\Delta T_3 = 0$ or $\Delta S = 0$ for "apparent" isospin and strangeness. Weak and strong interactions of pions might mix, of course, in some weak processes and, therefore, no rules for "apparent" isospin and strangeness would be observed in these processes.

The aim of the present note it to stress that the fields π^0 and π_0^0 (if the latter exists) have a charge parity $+1$ and -1,

$$(2) \qquad C\pi^0 C^{-1} = \pi^0, \qquad C\pi_0^0 C^{-1} = -\pi_0^0,$$

if the combinations $\Pi^0 \equiv \Pi_2 = (\pi^0 + i\pi_0^0)/\sqrt{2}$ and $\Pi^{0*} = \Pi_2^* = (\pi^0 - i\pi_0^0)/\sqrt{2}$ behave like fields of particles and antiparticles,

$$(3) \qquad C\Pi^0 C^{-1} = \Pi^{0*}.$$

Here, C denotes the unitary operator of charge conjugation. The proof is obvious, because we have

$$\pi^0 = \frac{1}{\sqrt{2}}(\Pi^0 + \Pi^{0*}), \qquad \pi_0^0 = \frac{1}{i\sqrt{2}}(\Pi^0 - \Pi^{0*}). \tag{4}$$

It is interesting to remark that, in case (3), the fields π^0 and π_0^0 are analogues of the fields K_1^0 and K_2^0 introduced by Gell-Mann and Pais [8] (Π^0 and Π^{0*} are analogues of K^0 and K^{0*}), but the roles of the fields in strong and weak interactions are interchanged.

An important consequence of the property (2) is the absence (parity and charge parity conserving) of Yukawa strong interactions between baryons and the second neutral pion π_0^0 (assuming π_0^0 is pseudoscalar like π^0), if only the corresponding strong interactions between baryons and the usual neutral pion π^0 are not forbidden. For instance, the strong interaction of π_0^0 with nucleons, $\bar{N}\gamma_5 N\pi_0^0$, is impossible.

In the theory proposed in [5] we can write

$$\pi^0 = \frac{1}{\sqrt{2}}(M_{12} + M_{21}), \qquad \pi_0^0 = \frac{1}{i\sqrt{2}}(M_{12} - M_{21}) \tag{5}$$

(and $\pi^+ = M_{11}$, $\pi^- = M_{22}$). Assuming relations

$$CM_{12}C^{-1} = M_{12}^* = M_{21} \quad \text{and} \quad CM_{21}C^{-1} = M_{21}^* = M_{12} \tag{6}$$

we have a theory with the property (3) (and therefore, also (2)), where

$$\Pi^0 = M_{12}, \qquad \Pi^{0*} = M_{21}.$$

Then, the hypothetical particle π_0^0 does not interact strongly (with conservation of parity and charge parity) with baryons in all cases in which the usual neutral pion π^0 does.

INSTITUTE OF THEORETICAL PHYSICS, WARSAW UNIVERSITY
(INSTYTUT FIZYKI TEORETYCZNEJ UNIWERSYTETU WARSZAWSKIEGO)
INSTITUTE FOR NUCLEAR RESEARCH, POLISH ACADEMY OF SCIENCES
(INSTYTUT BADAŃ JĄDROWYCH PAN)

REFERENCES

[1] B. d'Espagnat, J. Prentki and A. Salam, Nucl. Physics **3** (1957), 446, footnote on p. 449.

[2] A. M. Baldin, Nuovo Cimento **8** (1958), 569.

[3] A. M. Balding and P. K. Kabir, Nuovo Cimento **9**, (1958), 547.

[4] Y. Yamaguchu (preprint).

[5] W. Królikowski, Nucl. Physics **8** (1958), 461.

[6] W. Heisenberg, Proc. of Geneva Conference CERN (1958).

[7] W. Królikowski, Bull. Acad. Polon. Sci., Ser. des sci. mat., astr. et phys., **6** (1958), 417.

[8] M. Gell-Mann and A. Pais, Phys. Rev. **97** (1955), 1387.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 2, 1959

EXPERIMENTAL PHYSICS

On the Dependence of Transfer Phenomena in Semi-Conductors upon Magnetoconcentrational Effects

by

Z. KOPEĆ

Presented by A. SOŁTAN on December 9, 1958

Papers dealing with transfer phenomena in semi-conductors immersed in a magnetic field have, as yet, failed to account fully for the implications of the diamagnetism and paramagnetism of the electrons. In fact, only the effect of the magnetic field on electron kinetics has been taken into consideration by building the quantum theoretical conclusions from Landau's theory [1] into Boltzmann's transfer equation. However, a complete analysis of the effect of the magnetic field on the semi-conductor should involve accounting not only for the variations of the kinetic parameters, but also for those of carrier concentrations as resulting from the effect of the field on the energetical structure of the semi-conductor. In semi-conductors wherein the effective masses of the carriers are small, such variations in the band structure are found to be considerable; they have been the object of some direct experimental investigations carried out recently in the field of semi-conductor optics [2], [3]. These structural variations affect the concentration through:

1) the variation of the breadth of the forbidden band in the magnetic field,

2) the variation of the activation energy of the admixtures in the magnetic field accounting for the low-temperature anomaly of $\Delta\varrho/\varrho$ in InSb, and

3) the fine structure of the valence and conductivity bands themselves as well as that of the admixture levels, brought about by splitting.

The ensuing calculations show that experimental conditions may be found when the magnetoconcentrational effects are decisive for transfer in a semi-conductor immersed in a magnetic field. At weak fields, the effects from 3) are small as compared with those from 1) or 2), the last playing a part only at very low temperatures, whilst 1) is of importance

at temperatures corresponding to the transitory range and to the beginning of the intrinsic range (high temperatures).

Only this latter case will be analyzed here.

By Landau's theory, a magnetic field causes quantization of the energy of the electrons in the bands:

$$E(k,H)=\frac{\hbar^2 k}{2m^*_{\mp}}+\frac{e\hbar}{cm^*_{\mp}}\left(\frac{1}{2}+n\right)H, \tag{1}$$

wherein k is the wave vector in the field's direction,

H — the magnetic field intensity,

$m^*_{\mp}$ — the effective masses of the hole and the electron, respectively, and

n — a non-negative integer.

Hence, the breadth of the forbidden band is given by

$$\Delta E_G(H)=\Delta E_G(0)+\frac{1}{2}\frac{e\hbar}{c}\left(\frac{1}{m^*_{+}}+\frac{1}{m^*_{-}}\right)H \tag{2}$$

and the concentrations of the electrons and holes are field dependent:

$$\begin{aligned} n &= \frac{2(2\pi m^*_{-}kT)^{3/2}}{h^3}\frac{2}{\sqrt{\pi}}F_{1/2}\left(\frac{\zeta}{kT}\right) \\ p &= \frac{2(2\pi m^*_{+}kT)^{3/2}}{h^3}\frac{2}{\sqrt{\pi}}F_{1/2}\left(-\frac{\zeta}{kT}-\frac{\Delta E_G(H)}{kT}\right) \end{aligned} \tag{3}$$

with ζ denoting the Fermi energy level, and

$$F_{1/2}\left(\frac{\zeta}{kT}\right)=\int_0^{\infty}\frac{x^{1/2}\,dx}{1+\exp\left(x-\frac{\zeta}{kT}\right)}.$$

Fermi's integral: the concentrations n and p fulfill the condition of neutrality:

$$n-p=N-A, \tag{4}$$

where N and A are the concentrations of the ionized donors and acceptors, respectively.

Within the range of complete ionization, $N-A$ equals the number of carriers, which may be determined by measuring Hall's constant. Here, the solution of the system of Eqs. (2), (3) and (4) yields simple expressions for n and p which depend solely on the band structure and the field intensity. By this result, a field of 1000 g should produce a 1% variation in the electron concentration in InSb. A variation so considerable will take place only at sufficiently high temperatures (participation of intrinsic electrons in the conductions). The magnetoconcentrational effect, as assessed for two InSb samples of carrier concentration 10^{14} and 10^{17} cm^{-3}, is plotted in Fig. 1.

It should be stressed that, by this evaluation, the carrier concentration varies linearly with the intensity at weak fields, a result easily reached analytically; if classical statistics may be applied, the simple function $n(H)$ shown in Fig. 1 is obtained. Whenever concentrational effects are considerable, they will clearly affect all transfer phenomena. In accordance with the plot in Fig. 1, this will be the case especially in preferably pure p-type samples kept at sufficiently high temperatures corresponding to a steep slope of the $n(H)$ curve, but not high enough for the transfer phenomena to cease being concentration sensitive.

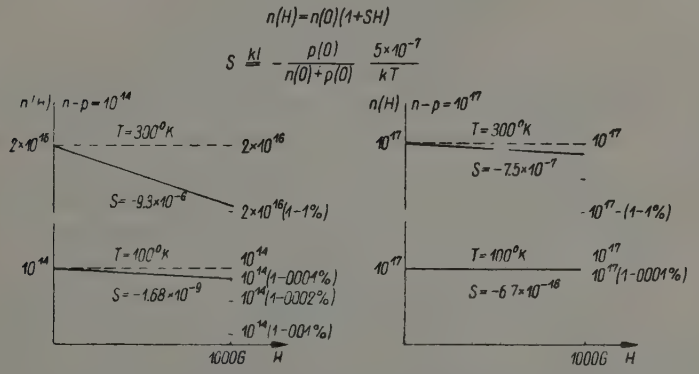


Fig. 1. Variation of concentration for 2 samples of InSb with $n-p = 10^{14}$ cm^{-3} and $n-p = 10^{17}$ cm^{-3} at temperature 100° K and 300° K

Some examples will now be considered in brief:

a) Weak magnetic fields.

To fix ideas, Hall's constant R will be considered (it would be equally possible to utilize α or any similar quantity):

$$R = \frac{r_+ - b^2 r_- x}{\mathrm{pec}(1+bx)}, \tag{5}$$

where $b = \frac{\mu_-}{\mu_+}$, $x = \frac{n}{p}$, and r_+, r_- are scattering integrals; the latter, like the quantity b, depend on the field intensity only through the kinetics. The dependence is a quadratic one, and is most evident in the case of the quantity b. Moreover, R varies in a magnetic field at the result of variations in the carrier concentration through the variations of $x(H)$ and $p(H)$. Quite generally,

$$\left(\frac{\Delta R}{R}\right) = \left(\frac{\Delta R}{R}\right)_{\mathrm{conc.}} + \left(\frac{\Delta R}{R}\right)_{\mathrm{scat}}. \tag{6}$$

The term accounting for scattering is of the form

$$(7)\qquad \left(\frac{\Delta R}{R}\right)_{\text{scat.}} = \left\{ \frac{\dfrac{\partial r_+}{\partial H} - 2b\dfrac{\partial b}{\partial H} r_- x - b^2 \dfrac{\partial r_-}{\partial H} x}{|r_+ - b^2 r_- x|} - \frac{2\partial b}{\partial H} \frac{x}{1+bx} \right\} H .$$

It is easily seen that $(\Delta R/R)_{\text{scat.}}$ assumes positive values already at the beginning of the transitory range and remains positive throughout the entire range of high temperatures; this term it proportional to H^2 (see e. g. [4]). On the other hand, the concentrational term

$$(8)\qquad \left(\frac{\Delta R}{R}\right)_{\text{conc.}} \doteq \frac{1}{R}\frac{\partial R}{\partial x}\frac{\partial x}{\partial H} H$$

is linear in a weak field, and thus at sufficiently weak intensities alone accounts for the variation arising from the field. The corresponding experimental range will be termed the range of concentrational approximation.

Eq. (8) should now be analyzed; $\partial x/\partial H$ is a numerical coefficient univocally connected with the slope of the straight lines in Fig. 1. By Eq. 2, for a given semi-conductor, the steepness it determined by $1/m_-^* + 1/m_+^*$ (for InSb, practically by $1/m_-^*$ alone). On the other hand, the factor $\partial R/\partial x$ depends solely, within the range of the same approximation, on the shape of the function $R(T)$, since here, small variations of the temperature effect the transfer phenomena solely through the variation in carrier concentration. Hence, a given variation ΔR may be obtained at constant temperature by applying a magnetic field, or at constant field intensity (e. g. at $H = 0$) by varying the temperature.

If Boltzmann statistics may be utilized, the following simple relationship it obtained:

$$(9)\qquad \frac{T - T_0}{T} = \frac{\dfrac{1}{2}\dfrac{e\hbar}{c}\left(\dfrac{1}{m_+^*} + \dfrac{1}{m_-^*}\right)}{kT\left(\dfrac{9}{2} + \dfrac{\Delta E_G}{KT} - \dfrac{1}{k}\dfrac{\partial \Delta E_G}{\partial T}\right)} H;$$

here, T_0 denotes the temperature at which the quantity characterizing the transfer phenomenon assumes a given value A (in the present case, $A = R(T_0)$) at zero field, and T is the temperature at which this quantity assumes the same value A at field intensity H.

By Eq. (9), $T > T_0$. Hence, a magnetic field may lead to a variation of the effect measured consisting in a decrease of the temperature at $H = 0$. If ΔE_G and $\partial \Delta E_G/\partial T$ are known, $1/m_-^* + 1/m_+^*$ may be determined from $(T - T_0)$ by Eq. (9). For InSb, substitution of the well known values of the parameters in Eq. (9) leads to $T - T_0 = 0.5°\text{C}$ at a field of 1000 Gauss (for Hall's effect, at the point of transition through zero).

This represents a value that is easily measured. Evaluation of the scattering parameters in Eq. (7) shows that the range of concentrational approximation is practically restricted to very weak fields. Allowing for maximum possible scattering effects, the following set of scattering and concentrational effects has been obtained with an InSb sample of rather high purity:

H (in oersted)	$\left(\frac{\Delta R}{R}\right)_{\text{conc.}}$	$\left(\frac{\Delta R}{R}\right)_{\text{scat.}}$	$\frac{\Delta R_{\text{conc.}}}{\Delta R_{\text{scat.}}}$
30	5×10^{-4}	2.5×10^{-5}	20
300	5×10^{-4}	2.5×10^{-3}	1
1000	1.5×10^{-2}	2.5×10^{-2}	0.60

Hence it is seen that, throughout the range of weak fields, the concentrational effects are considerable, and prevail over them due to scattering. However, the range throughout which the former prevail strongly does not exceed field intensities of the order of 100 Gauss, making it experimentally difficult to measure the purely concentrational part in effects such as that of Hall.

b) Range of medium and strong fields.

At medium fields, the concentrational effects still play an important part, although those due to scattering may be predominant. However, as in this range the kinetics is exceedingly involved and the formulae are not immediately clear, these effects will not be discussed in detail at present.

At strong fields, transfer phenomena once more present a simple picture. It will be remembered that, here, the processes of scattering tend toward saturation; this was the reason why the quantity R and others (e. g., α) were expected to tend to constant values R_∞ (or α_∞) with rising magnetic field intensities. However, even with the strongest fields, saturation was not observed experimentally [5]. It seems probable that these effects arise through the magnetoconcentrational effect.

Yet another experimental domain is of great interest, namely the one of longitudinal magnetic fields: $(\Delta\varrho/\varrho)_{\text{long.}}\,\alpha(H)_{\text{long.}}$, etc. In accordance with papers by Seitz, with bands of spherical structure (e. g., InSb is a semi-conductor of this kind) these quantities should present no variations resulting from scattering effects. However, it was found experimentally that

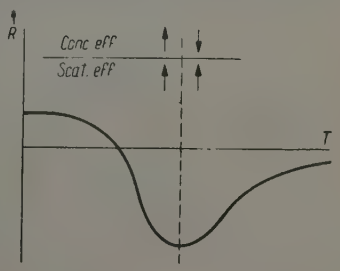


Fig. 2. Influence of magnetic field on the change of Hall coefficient effected by the concentration and scattering effects

a longitudinal field distinctly leads to variations of $(\Delta\varrho/\varrho)_{\text{long.}}$; the form of these variations, e. g. in InSb, constitutes an anomaly. It follows from the foregoing considerations that "pure" magnetoconcentrational effects should be expected to take place in longitudinal fields.

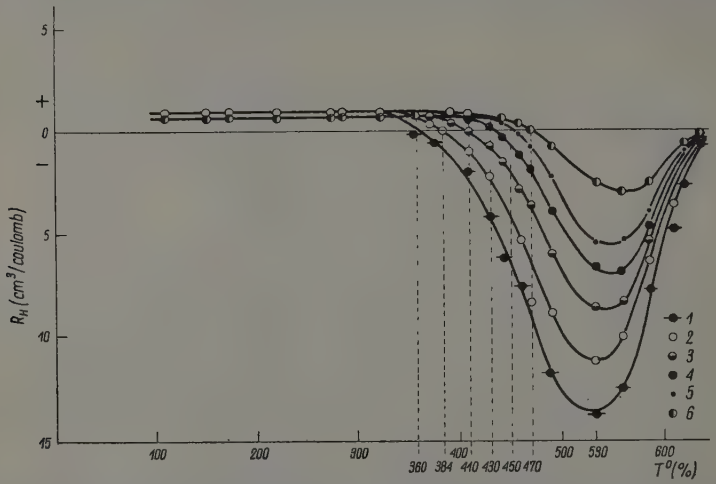


Fig. 3. [6] Dependence of Hall constant R on temperature and intensity of magnetic field for sample p-InSb : 1—1000 Gauss, 2—5000 Gauss, 3—10000 Gauss, 4—15000 Gauss, 5—20000 Gauss, 6—32500 Gauss

Owing to their simplicity and to the possibility of applying strong fields, analysis of these effects should yield highly interesting data concerning not only the linear variations (1) and (2), but also higher order variations (3) resulting from changes in the structure of the semi-conductor immersed in the field.

To illustrate the above considerations, experimental data on InSb recently published will be utilized. According to what has been said, the following superposition of concentrational and scattering effects should be expected to occur (Fig. 2).

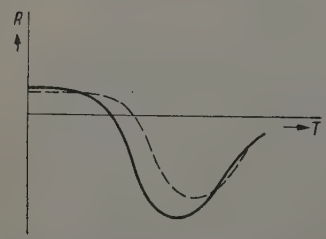


Fig. 4. Expected change of Hall coefficient in magnetic field in case of predominant influence of concentration effect

These expectations are corroborated by Nasledov and Khalidov's measurements [6] in fields which may be classified as medium and strong. To the left of the extremum, both effects should give rise to variations tending in the same direction, whilst to the right there should be opposite tendencies.

Fig. 3 fully confirms this conclusion.

It may also be seen from Fig. 2 that, to the right of the extremum and at moderate field intensities, the concentrational effects should predominate over those of scattering (near the inflexion point of the curve), as shown in Fig. 4.

This latter conclusion is, in turn, corroborated by the data recently published by Champness [7] and shown here in Fig. 5.

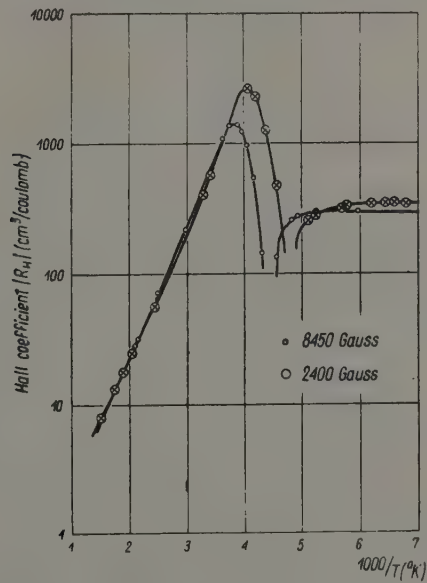


Fig. 5. [7] Hall coefficient variation with reciprocical temperature for sample p-InSb (2×10^{16}/cm^3) at two magnetic field strengths

The author wishes to thank Mrs P. Jaszczyn for her suggestions concerning the part played by variations of the concentration in a magnetic field.

INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT FIZYKI PAN)

REFERENCES

[1] L. Landau, Zt. f. Physik **64** (1930), 629.
[2] S. Zwerdling, Phys. Rev. **104** (1956), 1805.
[3] — ibid. **108** (1957), 1402.
[4] F. Seitz, Phys. Rev. **79** (1950), 372.
[5] P. Kapitza, Proc. Roy. Soc. **123** (1929), 291.
[6] D. N. Nasledov and L. O. Khalidov, Zh. Techn. Fiz. **26** (1956), 8.
[7] C. H. Champness, Journ. of Electronics **4** (1958), 201.

БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VII 1959 ВЫПУСК 2

М. К. ФОРТ, ε-ОТОБРАЖЕНИЯ ДИСКА НА ТОРУС . . . . стр. 51–54

Пусть D — обозначает единичный диск в евклидовой плоскости, а T — поверхность торуса. Существует $\varepsilon>0$ такое, что для каждого отображения непрерывного f диска D на T, по крайней мере, одно из множеств $f^{-1}(y)$, где $y \in T$, имеет сечение $>\varepsilon$. Этот результат является решением проблемы, поставленной С. Улямом в так. наз. „Шотландской Книге".

Г. АЛЬБРЫХТ, НЕКОТОРЫЕ ЗАМЕЧАНИЯ О ПРОСТРАНСТВЕ МАРЦИНКЕВИЧА-ОРЛИЧА. III. . . . . стр. 55–56

В настоящем сообщении показано, каким образом можно построить линейное, метрическое и полное пространство — аналогичное пространству Марцинкевича-Орлича — элементами которого являются классы функций определенных на произвольной, бесконечной группе.

В. ПОГОЖЕЛЬСКИЙ, О НЕКОТОРЫХ КЛАССАХ КОМПЛЕКСНЫХ ФУНКЦИЙ, ОПРЕДЕЛЕННЫХ НА РАЗОМКНУТЫХ ДУГАХ . стр. 57–62

В работе автором был определен класс $\mathfrak{H}_a^\mu$ комплексных функций $\varphi(t)$, заданных на разомкнутых дугах $\widehat{a_1b_1}, \widehat{a_2b_2}, \ldots, \widehat{a_pb_p}$, при помощи неравенств (1) и (2). Затем, автор представил свойства этих классов в четырех теоремах. Первая теорема устанавливает, что преобразование функции класса $\mathfrak{H}_a^\mu (\alpha>0)$ при помощи сингулярного интеграла Коши (4) дает функцию того же класса. Вторая теорема устанавливает, что класс $\mathfrak{H}_a^\mu$ является частью класса $\mathfrak{H}_{a_1}^{\mu_1}$, если $0 \leqslant \alpha \leqslant \alpha_1$, $0 < \mu_1 \leqslant \mu$. Третья теорема устанавливает, что если функция $\varphi(\tau)$ принадлежит к классу $\mathfrak{H}_a^\mu(c_{k_1}, \ldots, c_{k_q})$, т. е. она удовлетворяет неравенствам (8) и (9), то функция $\psi(t)$, определенная формулой (10) принадлежит к классу $\mathfrak{H}_\varrho^\mu(c_{k_1}, \ldots, c_{k_q})$. Четвертая теорема устанавливает, что если функция $\varphi(t)$ принадлежит к классу $\mathfrak{H}_a^\mu(c_{k_1}, \ldots, c_{k_q})$, то функция, определенная формулой (11), ограничена в окрестности каждой точки $c_{k_1}, \ldots, c_{k_q}$ и имеет оценку (12) в окрестностях других концов дуг $\widehat{a_\nu b_\nu}$.

Ю. ПЭРКАЛЬ, ДВА НОВЫХ МЕТОДА АНАЛИЗА БАТЕРИИ ПРИЗНАКОВ . . . . . стр. 63—66

В работе представлены два новых метода анализа батерии признаков: одновекторный и многовекторный. Эти методы аналогичны методам факторного анализа, а именно: монофакторному методу (Спирмана) и многофакторному методу (Терстона). Вместо факторов ортогональных между собой и некоррелированных с остатками, автор предлагает векторы полученные путем линнейных исчислений из исходных признаков. Векторы могут быть коррелированы так между собой, как и с остатками. В отличие от факторных методов, предложенные методы являются численно простыми и однозначными, причем векторы имеют понятный природный смысл.

Г. ШТЕЙНГАУЗ и Ст. ТРЫБУЛА, О ПАРАДОКСЕ ВЕРОЯТНОСТЕЙ ПРИМЕНЕНИЯ . . . . . стр.

Если x, y, z обозначают три независимые случайные переменные, то может случиться, что каждое из трех неравенств

$$x < y, \quad y < z, \quad z < x$$

выполняется с вероятностью бóльшей, чем $\frac{1}{2}$.

Этот парадокс делает затруднительным сопоставление партий товара путем сравнивания попарно экземпляров, принадлежащих к двум партиям; этого парадокса не устраняют вычисления ожидаемого значения и среднего отклонения.

З. ОПЯЛЬ, О СУЩЕСТВОВАНИИ ПЕРИОДИЧЕСКИХ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА . . . . стр. 71—75

Автор рассматривает дифференциальное уравнение второго порядка

$$u'' = f(t, u, u')$$

при предположении, что функция $f(t, u, v)$ достаточно регулярна, не убывающая по отношению к u и периодическая по отношению к t с периодом 1. При таких предположениях автор доказывает, что существование ограниченного интеграла вместе с производной в интервале $(0, +\infty)$ влечет за собой существование периодического интеграла с периодом 1.

И. БЯЛЫНИЦКИЙ-БИРУЛЯ, О МОДЕЛИ ЛИ . . . . . стр. 77—83

В работе показано, что ренормализация в модели Ли и возникающие в связи с этим вопросы тесно связаны с устранением половины членов из Гамильтониана взаимодействия. В случае простой модели закрепленных фермионов учет всех членов устраняет все трудности. В таком случае нет необходимости введения индефинитной метрики.

Б. КУХОВИЧ, ПОЛЯРИЗАЦИЯ ЭЛЕКТРОНОВ В ПЕРВЫХ ЗАПРЕЩЕННЫХ УНИКАЛЬНЫХ ПЕРЕХОДАХ β . . . . . стр. 85—91

В работе даются формулы для угловой корреляции электрон-нейтрино и поляризации электронов в первых запрещенных уникальных β-переходах с изменением спина $\Delta J = \pm 2$. Применялся метод Катковского-Делоффа для вычислеиия квадратов элементов матрицы перехода при помощи техники шпуров. Вычисления проведены точно с учетом всех кулоновских эффектов и возможных интерференций между аксиально-векторным и тензорным взаимодействиями.

Приводятся асимптотические значения для $\frac{\alpha Z}{p} \ll 1$, $p\varrho \ll 1$ (где p — обозначает момент электрона, а ϱ — ядерный радиус).

К. Л. НОДЖ и Я. ЖЕВУСКИЙ, ОБ ЭКВИВАЛЕНТНОСТИ НЕКОТОРОГО КЛАССА НЕ-ЛОКАЛЬНЫХ ТЕОРИЙ ПОЛЯ И ТЕОРИЙ СОДЕРЖАЩИХ ПРОИЗВОДНЫЕ ПОЛЕЙ ВЫСШЕЙ СТЕПЕНИ . . . . стр. 93—96

Полагая особый случай условия Боголюбова для элиминирования не физических следствий теории с неограниченной метрикой, авторы доказывают, что некоторый класс не-локальных теорий эквивалентен теориям, описанным при помощи локального Лагранжиана, содержащего производные полей высшей степени.

В. КРУЛИКОВСКИЙ, О ВТОРОМ НЕЙТРАЛЬНОМ МЕЗОНЕ π . . . . . стр. 97—98

В работе автор проводит дискуссию относительно зарядной четности гипотетического второго нейтрального мезона π и указывает на возможность отсутствия сильных воздействий для этой частицы.

З. КОПЕТЬ, ВЛИЯНИЕ МАГНИТОКОНЦЕНТРАЦИОННЫХ ЭФФЕКТОВ НА ЯВЛЕНИЯ ТРАНСПОРТА В ПОЛУПРОВОДНИКАХ стр. 99—105

Доказывается, что для полупроводников с малыми эффективными массами, как InSb, в случае сравниваемых концентраций носителей обоих знаков, магнитное поле приводит к сильному изменению этих концентраций. При благоприятных условиях (чистый образец, оптимальная температура), в поле с индукцией 1000 гауссов, эти изменения для InSb достигают 1%. Рассматриваемые эффекты обусловлены влиянием магнитного поля на структуру зон полупроводника (прежде всего — изменение ширины запретной зоны); в области слабых полей $\left(\frac{\mu H}{c} < 1\right)$ они зависят линейно от поля и во всех гальвано- и термомагнитных явлениях преобладают над эффектами рассеяния (пропорциональными H^2) вплоть до некоторых значений H_1. Во всей этой области влияние поля равносильно влиянию изменения температуры на концентрацию. Результаты оценки магнитоконцентрационных эффектов сравниваются, наконец, с опубликоваиными экспериментальными данными, подтверждающими вышесказанное.